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Current Matrix Elements from a Relativistic Quark Model*

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A relativistic equation to represent the symmetric quark model of hadrons with harmonic interaction is used to define and calculate matrix elements of vector and axial-vector currents. Elements between states with large mass differences are too big compared to experiment, so a factor whose functional form involves one arbitrary constant is introduced to compensate this. The vector elements are compared with experiments on photoelectric meson production, K_{l3} decay, and $\omega \to \pi\gamma$. Pseudoscalar-meson decay widths of hadrons are calculated supposing the amplitude is proportional (with one new scale constant) to the divergence of the axial-vector current matrix elements. Starting only from these two constants, the slope of the Regge trajectories, and the masses of the particles, 75 matrix elements are calculated, of which more than $\frac{3}{4}$ agree with the experimental values within 40%. The problems of extending this calculational scheme to a viable physical theory are discussed.

INTRODUCTION

The symmetric, nonrelativistic harmonic-oscillator quark model has been shown by a number of people[1,2] to offer considerable promise of helping to organize the wealth of data in the resonance region for high-energy phenomena. We intend here to bring some of these results together in a unified method of calculation in order to judge better the validity of this organizing power.

A truly relativistic quantum-mechanical theory today seems available only in the complexities of field theory with its many virtual states involving, for example, pairs, etc. It is so complex that no particular dynamic regularities among the resonances are expected of it, other than those resulting from symmetries of the original Hamiltonian. We have gone in a different direction, sacrificing theoretical adequacy for simplicity. We shall choose a relativistic theory which is naive and obviously wrong in its simplicity, but which is definite and in which we can calculate as many things as possible – not expecting the results to agree exactly with experiment, but to see how closely our "shadow of the truth" equation gives a partial reflection of reality. In our attempt to maintain simplicity, we shall evidently have to violate known principles of a complete relativistic field theory (for example, unitarity). We shall attempt to modify our calculated results in a general way to allow, in a vague way, for these errors.

This is, of course, quite dangerous – because if one allows too much latitude in modifying the results of the calculations, especially if empirical results are allowed to influence strongly the many arbitrary choices, the significance of later partial agreement with experiment is compromised and hard to interpret. We have kept our choices down to three constants in trying to fit over 100 data. The first of these constants is the spacing of levels per unit of angular momentum along trajectories (that is, the reciprocal slope of the Regge trajectories), chosen to be the same for baryons and mesons, $\Omega = 1.05$ GeV^2. The second constant is a measure of the strength of the pseudoscalar-meson coupling to hadrons. With just these, we find that our theoretical matrix elements drift further and further afield as the masses of the resonances increase. We interpret this as a consequence of the lack of unitarity and, in a most unsatisfactory way, have included in all matrix elements an adjustment factor F, a Gaussian in the center-of-mass momentum of the reaction, to cut the calculated elements down at higher energies. This is frankly just empirical fitting – the Gaussian form is chosen by analogy to such a factor in the nonrelativistic harmonic oscillator. The coefficient in the exponential of the Gaussian is our third constant. Our philosophy is to maintain simplicity and calculate everything directly and in first order, as it were. No allowances will be made for configuration mixing, spin-orbit coupling, and other physically real complications, for we do not mean that our original hypotheses are so near the truth to warrant such modifications, rather than modifications at the very beginning of our naive assumptions. The point of all this is to get some idea whether there is any resemblance to the real situation in the quark harmonic-oscillator model. We have concluded from our analysis that there is indeed. In the following sections we present the details of the calculations and the results.

HADRON SPECTRUM

The states in the harmonic-oscillator quark model are characterized as follows. For the baryons of three quarks we have, besides the spin and unitary-spin multiplets $\underline{56}$, $\underline{70}$, and $\underline{20}$, excitations of two independent three-dimensional modes of internal harmonic oscillation among the three particles. For the lower excitations, a state can be uniquely described by giving the total N, the total number of excitations of all the modes, and the L or total orbital angular momentum. For higher states this is not unique (there may be several ways to make up the same total L from the two angular momenta of the two three-dimensional internal degrees of freedom). We shall write each SU_6 multiplet state as $[\underline{A}, L^P]_N$, where $\underline{A}$ is $\underline{56}$, $\underline{70}$, or $\underline{20}$, L is the total orbital angular momentum, N the total quantum number, and the parity $P=(-1)^N$. If we further wish to specify an SU_3 multiplet within this, we adjoin ${}^d(\underline{B})_J$, where $\underline{B}$ is the SU_3 multiplet which must be $\underline{1}$, $\underline{8}$, or $\underline{10}$ (since singlet, octet, or decimets are all we can expect), d is the spin multiplicity (four for total of quark spins $\frac{3}{2}$, two for total quark spin $\frac{1}{2}$), and J is the total angular momentum of the state.

If we wish to speak of a particular particle within a multiplet, we finally add its name. Thus the neutron is $N(938)\ {}^2(\underline{8})_{1/2}[\underline{56}, 0^+]_0$. The $\underline{56}$ contains ${}^2(\underline{8})$ and ${}^4(\underline{10})$, the $\underline{70}$ is ${}^2(\underline{1})$; ${}^2(\underline{8})$; ${}^4(\underline{8})$; ${}^2(\underline{10})$; and the $\underline{20}$ is ${}^4(\underline{1})$; ${}^2(\underline{8})$. We have not yet identified any states belonging to the $\underline{20}$.

The multiplets we expect from the model, and the observed highest hypercharge state of each multiplet (N for octets, Δ for decimets, Λ for singlets) are given in Table II below. The other states of the multiplets are identified where known (as "accepted" in the Particle Properties Tables[3]) in Table IV below.

The numbers after each state are the mass squared. In this variable we have noticed some regularities.[4] First, of course, is the Regge relation, that adding $L=2$ repeats the multiplet $2\Omega = 2.10$ GeV^2 higher (e.g., compare $[\underline{56}, 0^+]_0$ and $[\underline{56}, 2^+]_2$, or $[\underline{70}, 1^-]_1$ and $[\underline{70}, 3^-]_3$). Masses seem to depend on L, but there is little spin-orbit coupling; each J has the same energy for given L. There is one enigmatic outstanding exception: $\Lambda(1405)\ {}^2(\underline{1})_{1/2}$ and $\Lambda(1520)\ {}^2(\underline{1})_{3/2}$. Inside the multiplets we would expect Σ and Λ to be degenerate. If we take the average of the Σ and Λ mass squared as the mass corresponding to strangeness $S=-1$ for an octet, we can see that the mass square rises by about 0.45 for each unit of $-S$ in octets, and by about 0.40 in decimets. (Here and in the following we measure mass squares in GeV^2.) Only the $\Sigma(1915)$ does not fit well into its octet, and fits even worse into the decimet at $J=\frac{5}{2}^+$. There are strong differences in mass squared depending on the spin relation (quartet or doublet) of the quarks and the unitary-spin relation ($\underline{1}$, $\underline{8}$, or $\underline{10}$). It does look as if the energy can be nearly separated into a sum of terms: one for strangeness (described above), one for SU_6-multiplet character,

$$\begin{array}{lll} \underline{56} & {}^2(\underline{8}) & 0.88 \\ & {}^4(\underline{10}) & 1.55 \\ \underline{70} & {}^2(\underline{1}) & 1.25 \text{ (or 0.92)} \\ & {}^2(\underline{8}) & 1.30 \\ & {}^4(\underline{8}) & 1.80 \\ & {}^2(\underline{10}) & 1.70, \end{array} \qquad (1)$$

and one for orbital energy, $1.05N$ plus possible additional corrections for the $N=2$ state, depending on how the two orbits are compounded,

$$
\begin{aligned}
N=2:\quad & 0_s=-0.82,\\
& 2_s=0,\\
& 2_{\alpha,\beta}=+0.06, \qquad\qquad (2)\\
& 0_{\alpha,\beta}=-0.23,\\
& 1_A \text{ unknown.}
\end{aligned}
$$

The exact meaning of these last symbols is made precise in the Appendix. We noticed empirically that the $\Sigma^2-\Lambda^2$ mass-square difference may alternate in sign with parity, being about $0.16\times$parity – again excepting the $\Sigma(1915)$, or supposing it to belong to the decimet (for evidence that this may be the case, see below). Such rules give all the masses to within 1.5% except for the $\Lambda(1405)$ enigma and the $\Sigma(1915)$.

Our model will be a harmonic oscillator whose eigenvalues are m^2. This will describe the m^2 varying as $N\Omega$, but will not take into account what spin-orbit energy variation there really is, as well as the corrections (2). We shall simply suppose that, without disturbing the form of the Hamiltonian otherwise, a constant, C, is added. This C gets a contribution (around 0.4 GeV^2) for each strange quark. Less satisfactory, the internal splits in the multiplet (1) are given no explanation, but are again represented by just adding the constants (1) to the value of C.

For mesons, the theoretical classification of the states of q and $\bar{q}$ is easier. There is just one SU_6 multiplet $\underline{36}$, so we will not put that in our notation. It consists of a spin-0 octet and singlet (or "nonet"), and a spin-1 nonet. These we call singlet and triplet states, respectively. The internal angular momentum combines with this spin to make a total J. The experimental situation among the mesons is more confused than among the baryons, and we have only chosen to try to identify the states of $N=0$ and some of $N=1$. We therefore need no elaborate notation, and use ${}^1S_0$ and ${}^3S_1$ for the $N=0$ singlet and triplet nonets; ${}^3P_2$, ${}^3P_1$, and ${}^3P_0$ for the $N=1$ states made from the triplet combined with $L=1$ to make $J=2$, 1, and 0; and ${}^1P_1$ for the singlet combined with $L=1$ to make $J=1$. The S states have negative parity, the P states positive parity. We have identified the states as given in Table V below.

Spin-orbit coupling is more obvious than in the baryons, but we continue to ignore it in our "Hamiltonian for m^2." We suppose that in triplet nonets, states like ω and ϕ are mixed so that ϕ and f' are made of pure strange quarks. For singlet nonets like η and η', we suppose there is no mixing. Again, we suppose that on trajectories m^2 rises by $1.05L$ [for example, $\rho_N(1670)$ 2.77 is probably 3^-, a repetition of $\rho(765)$ 0.58, and possibly degenerate with the trajectory through the ${}^3P_2\ A_2$ near $m^2=1.69$]. We have no explanation for the apparent splitting of the A_2 and consider the split peak to be one resonance (of width near 90 MeV).

For S states, each strange quark supplies 0.24 GeV^2 to the mass squared. The ϕ has two strange quarks, but η and η' are not diagonal in this quantity; the mean number is $\frac{4}{3}$ for η and $\frac{2}{3}$ for η'. Something must be assumed to push the SU_6 singlet (unitary singlet and spin singlet) η' up in energy.

BARYON DYNAMICS

Our point of departure is the nonrelativistic harmonic-oscillator symmetric quark model of Greenberg.[2] This raises a number of points which we have tried to solve. The main difficulty is that the excitation energies of the states are not small enough relative to the masses themselves to warrant using the nonrelativistic principles. This is especially true for mesons. It also leaves open the question of the quark mass, whether it is $\frac{1}{2}$ the pion mass or $\frac{1}{3}$ the nucleon mass; for harmonic excitations can only lead to energies above the rest mass of the parts.

A simple harmonic-oscillator Hamiltonian is

$$H=\frac{1}{2m}P^2+\tfrac{1}{2}m\omega_0{}^2X^2.$$

Multiply by $2m$ and set $m^2\omega_0{}^2=\Omega^2$, to get

$$2mH=P^2+\Omega^2X^2,$$

and the quark mass disappears on the right-hand side.

But what is the left-hand side? If we add a constant, m^2, it is m^2+2mH or its eigenvalues, m^2+2mW, representing approximately the squares of the relativistic energies $(m+W)^2$ for small W. With fixed Ω^2, we will now find the mass squared of the states rising linearly with energy, and no quark mass to decide upon.

With this clue, we are led, in the example of three quarks in interaction, to consider an operator[5] (we discuss spin later)

$$K=3(p_a{}^2+p_b{}^2+p_c{}^2)+\tfrac{1}{36}\Omega^2[(u_a-u_b)^2+(u_b-u_c)^2+(u_c-u_a)^2]+C, \qquad (3)$$

where C is a constant. (The 3 and 36 are scales chosen to simplify later expressions.) Here, $p_a{}^2$ represents the square of the four-vector of the momentum operator of quark a,

$$
\begin{aligned}
p_a{}^2&=p_{a\mu}p_{a\mu}\\
&=p_{at}\,p_{at}-p_{ax}p_{ax}-p_{ay}p_{ay}-p_{az}p_{az}.
\end{aligned}
$$

The conjugate position is $u_{a\mu}$, so $p_{a\mu} = i\partial/\partial u_{a\mu}$. We shall suppose that the propagator for baryons is $1/K$. If there are perturbations $-\delta K$ in K due to external fields, etc., we should have $(K-\delta K)^{-1}$ for the propagator. This is

$$\frac{1}{K}+\frac{1}{K}\delta K\frac{1}{K}+\frac{1}{K}\delta K\frac{1}{K}\delta K\frac{1}{K}+\text{etc.}$$

In our example, K is separable, and the external momentum of the entire state, $P=p_a+p_b+p_c$, can be separated from the internal motion (which, in turn, can be separated further into normal-mode oscillators). Let

$$\begin{aligned} p_a &= \tfrac{1}{3}P - \tfrac{1}{3}\xi, \\ p_b &= \tfrac{1}{3}P + \tfrac{1}{6}\xi - \frac{1}{2\sqrt{3}}\eta, \\ p_c &= \tfrac{1}{3}P + \tfrac{1}{6}\xi + \frac{1}{2\sqrt{3}}\eta, \end{aligned} \tag{4a}$$

with the coordinate operators

$$\begin{aligned} u_a &= R - 2x, \\ u_b &= R + x - \sqrt{3}y, \\ u_c &= R + x + \sqrt{3}y, \end{aligned} \tag{4b}$$

where R, x, and y are conjugate to the momenta P, ξ, and η. Then

$$K = P^2 - \mathfrak{M}, \tag{5}$$

where $\mathfrak{M}$, which we shall call the mass-square operator, depends only on the internal motion and is

$$-\mathfrak{M} = \tfrac{1}{2}\xi^2 + \tfrac{1}{2}\eta^2 + \tfrac{1}{2}\Omega^2 x^2 + \tfrac{1}{2}\Omega^2 y^2 + C, \tag{6}$$

two harmonic oscillators, each giving eigenvalues spaced by Ω.

The propagator between disturbances $1/K$ now becomes $(P^2-\mathfrak{M})^{-1}$, and it may be written in terms of eigenfunctions $h_i(\xi,\eta)$ and eigenvalues $\mathfrak{M}_i$ of the operator $\mathfrak{M}$ in the form

$$\frac{1}{P^2-\mathfrak{M}} = \sum_i h_i(\xi,\eta)\frac{1}{P^2-\mathfrak{M}_i}\bar{h}_i(\xi,\eta), \tag{7}$$

where $\bar{h}$ is adjoint to h in a sense defined below. The quantity P propagates from disturbance to disturbance as a constant, changed at each disturbance by the total momentum brought in by the disturbance. Putting this together and looking at the poles, we see that we are representing the propagation of an object having one or another value of the square of the mass,

$$m_i^{\,2} = \mathfrak{M}_i, \tag{8}$$

given by the eigenvalues of $\mathfrak{M}$, and the amplitude that a disturbance makes a change from propagation of the state i with mass m_i to propagation of the state j with mass m_j is simply the matrix element

$$N_{ji} = \langle \bar{h}_j(\xi,\eta)|\delta K|h_i(\xi,\eta)\rangle. \tag{9}$$

This, being the matrix element of perturbations in eigenvalues of P^2, is what goes directly, for N, into the usual relativistic rate formula for two-body decay of a system of mass m_1 into parts of momentum Q,

$$\Gamma = \frac{Q}{8\pi m_1^{\,2}}\int |N|^2\frac{d\Omega}{4\pi}. \tag{10}$$

Evidently, since $\mathfrak{M}$ is like the Hamiltonian of an harmonic oscillator, its eigenvalues will be a succession of integers N times Ω, so the mass squared can grow linearly with the angular momentum. We shall leave into C those energies for which we have not developed a dynamical explanation. For example, it must contain a term for each strange quark, and some spin and unitary-spin exchange terms to generate the numbers (1).

We have, however, the serious difficulty that we have oscillators in four dimensions and the excitation of the timelike mode gives trouble. For example, the lowest state (ground-state function in momentum space) is

$$\begin{aligned} h_0(\xi,\eta) &= \exp[(\xi^2+\eta^2)/2\Omega] \\ &= \exp\{[(p_a-p_b)^2+(p_b-p_c)^2+(p_c-p_a)^2]/\Omega\}. \end{aligned} \tag{11}$$

The space parts of $\xi^2 = \xi_t^{\,2} - \xi_x^{\,2} - \xi_y^{\,2} - \xi_z^{\,2}$ present no trouble, being expected from the Gaussian wave functions for the nonrelativistic harmonic oscillator. However, the time variable leads to meaningless integrals. We see this difficulty again if we write $\mathfrak{M}$ in the following more formal fashion. Let a_μ^* and a_μ be a set of creation and annihilation operators related to the variables ξ_μ as follows:

$$\xi = \left(\frac{\Omega}{2}\right)^{1/2}(a^*+a), \quad x = -i\left(\frac{1}{2\Omega}\right)^{1/2}(a^*-a); \tag{12a}$$

and likewise put

$$\eta = \left(\frac{\Omega}{2}\right)^{1/2}(b^*+b), \quad y = -i\left(\frac{1}{2\Omega}\right)^{1/2}(b^*-b), \tag{12b}$$

with the commutators $[a_\mu, a_\nu^*] = [b_\mu, b_\nu^*] = -\delta_{\mu\nu}$, where $\delta_{\mu\nu}$ equals -1 for space, $+1$ for time variables. We can write

$$\mathfrak{M} = -\Omega(a_\mu^* a_\mu + b_\mu^* b_\mu) + C \tag{13}$$

and suppose that the excited states are given by multiplication of h_0 by various powers of a^* or b^*. The first-excited states of the oscillator ξ would then be $a_x^*h_0$, $a_y^*h_0$, $a_z^*h_0$, and $a_t^*h_0$. When we take sums of states for tests of unitarity, etc., we shall need the adjoints of these, h_0a_x, h_0a_y, h_0a_z, and $-h_0a_t$. The latter is such that in taking norms, invariant expressions such as $h_0(a_xa_x^* + a_ya_y^* + a_za_z^* - a_ta_t^*)h_0$ will arise. Thus the rule for forming $\bar{h}$, adjoint from $h = (\text{oper.})h_0$, is to take the complex conjugate and change each a_μ^*, b_μ^* in the operator to a_μ, b_μ if μ is a space index, and to $-a_\mu$, $-b_\mu$ if it is a time index. The time states can have negative norm (and positive energy).

We shall suppose that only the spacelike excited states exist and compute matrices such as N_{ji} only for those i and j which involves space excitation. The criterion is that only those states, $|h\rangle$, are to be used that satisfy $(P\cdot a)|h\rangle = (P\cdot b)|h\rangle = 0$. Our simple theory will then not be really able to deal with nature in a complete way. For example, there are a number of sum rules for the electric current. For them to be satisfied, a relation like

$$\sum_j (k|e^{iq_1\cdot x}|j)(j|e^{iq_2\cdot x}|l) = (k|e^{i(q_1+q_2)\cdot x}|l) \tag{14}$$

must be satisfied, and is satisfied if we sum properly over *all* states. To see the kind of error we make in leaving out the time states, choose $q_1 = -q_2$, $k = l = 0$, say, to get

$$\sum_{\text{all } j} (0|e^{iq\cdot x}|j)(j|e^{-iq\cdot x}|0) = 1.$$

Now $(0|e^{iq\cdot x}|j)^* = (-1)^{n_j}(j|e^{-iq\cdot x}|0)$, where n_j is the number of timelike excitations in the state j. Thus, if we restrict ourselves to spacelike states only, our sum

$$\sum_{j=\text{space only}} |(0|e^{iq\cdot x}|j)|^2$$

will exceed unity, because the time-state contributions have not been subtracted away.

Thus our matrix elements could be expected to be too large, and so they turn out, in comparison with experiment, to be. To remedy this in a purely *ad hoc* and arbitrary fashion, we have chosen to multiply each matrix by an adjustment factor analogous to the $\exp(-\text{const}\ \vec{Q}^2)$ of the nonrelativistic oscillator. This is to replace a factor $\exp[\text{const} \times (q_t^2 - \vec{Q}^2)]$ which the relativistic oscillator gives.

We have chosen the form

$$F = \exp\left(\frac{-M_1^2Q^2}{\Omega(M_1^2 + M_2^2)}\right) \tag{15}$$

for the adjustment factor for a matrix element of a junction of three lines of four-momenta P_1, P_2, and P_3, so $P_1 + P_2 + P_3 = 0$. Here $M_1^2 = P_1^2$, etc., and $\vec{Q}$ is the space momentum of one of the particles 2 or 3, in the system at rest with respect to particle 1. The combination $M_1^2Q^2$ is a symmetrical invariant,

$$\begin{aligned} M_1^2Q^2 &= (P_1\cdot P_2)^2 - P_1^2P_2^2 = (P_2\cdot P_3)^2 - P_2^2P_3^2 \\ &= (P_3\cdot P_1)^2 - P_3^2P_1^2 \\ &= \tfrac{1}{4}(M_1^4 + M_2^4 + M_3^4 - 2M_1^2M_2^2 \\ &\qquad - 2M_2^2M_3^2 - 2M_3^2M_1^2). \end{aligned} \tag{16}$$

The denominator $M_1^2 + M_2^2$ was chosen arbitrarily to try to imitate the lack of dependence of the coefficient of Q^2 on the mass of the state in the nonrelativistic case. Nothing is sensitive to this precise form of the denominator; a constant would probably do as well with a newly adjusted coefficient to replace $1/\Omega$. We have chosen the coefficient $1/\Omega$ to be the same for hadrons and mesons, to fit the data. (We do not have a theoretical argument for this equality and may have allowed ourselves an independent choice for mesons if the data warranted it. Therefore, strictly we should say that we have two adjustable parameters in these factors, one for hadrons, one for mesons. The results are not at all sensitive to these choices.)

The quarks have spin $\frac{1}{2}$ and our expression (3) for K appears not to involve spin. We do not desire any complicated involvements in view of the apparent lack of a strong spin-orbit coupling in the baryons. This we can arrange by interpreting p_a^2 as the square of the Dirac operator $\not{p}_a\not{p}_a = (p_{a\mu}\gamma_\mu) \times (p_{a\mu}\gamma_\mu)$. This changes the equation in no way – it only defines the perturbational effects of electromagnetic potentials. When an electric potential A_μ is present, we are to replace $\not{p}_a$ by $\not{p}_a - e_a\not{A}(u_a)$, so the first-order perturbation in such a potential is

$$\delta K = 3\sum_\alpha e_\alpha[\not{p}_\alpha\not{A}(u_\alpha) + \not{A}(u_\alpha)\not{p}_\alpha], \tag{17}$$

where the sum on α is over the quarks a, b, and c, and where e_a is the charge operator on quark a (e.g., $+\frac{2}{3}$ if a is of type u, $-\frac{1}{3}$ if of type s or d).

Thus, for the interaction with a wave of polarization vector, e_μ, and momentum carried in, q_μ, we have the interaction, $j_\mu^V e_\mu$, where

$$j_\mu^V = 3\sum_\alpha e_\alpha(\not{p}_\alpha\gamma_\mu e^{iq\cdot u_\alpha} + \gamma_\mu e^{iq\cdot u_\alpha}\not{p}_\alpha). \tag{18}$$

(In second order there is an extra coupling from the term $-\sum_\alpha 3e_\alpha^2[A(u_\alpha)\cdot A(u_\alpha)]$.)

For the axial-vector current, we need merely replace γ_μ by $i\gamma_5\gamma_\mu$,

$$j_\mu^A = 3i\sum_\alpha e'_\alpha(\not{p}_\alpha\gamma_5\gamma_\mu e^{iq\cdot u_\alpha} + \gamma_5\gamma_\mu e^{iq\cdot u_\alpha}\not{p}_\alpha), \tag{19}$$

as can be seen by starting from the Dirac equation in an axial-vector potential B_μ, $(\not{p} - \gamma_5\not{B})\psi = m\psi$, and squaring it to get an equation for m^2,

$$m^2\psi = (\not{p} - \gamma_5\not{B})(\not{p} - \gamma_5\not{B})\psi \ .$$

Thus the rule for axial-vector currents is to replace $\not{p}_a$ by $\not{p}_a - e_a'\gamma_5\not{B}$, where e_a' is the axial charge operator of the quark (depending on which unitary component of axial-vector current is wanted). The divergence of this current is evidently

$$-iq_\mu j_\mu^A = 3\sum_\alpha e_\alpha'(\not{p}_\alpha\gamma_5\not{q}e^{iq\cdot u_\alpha} + \gamma_5\not{q}e^{iq\cdot u_\alpha}\not{p}_\alpha). \qquad (20)$$

Now, again we have too many states and shall have to make some restrictions. The unperturbed K, expression (3), does not involve γ_μ and therefore does not tell us the direction in which the four-component spinors of the quarks must point. That is fine for the two spin components, for we do not wish to couple the spin direction. But neither does anything govern the size of what are often called the small components. How do we know that we have, properly, the spinor of a quark particle and not of an antiquark? In Dirac theory, ψ satisfies $p^2\psi = m^2\psi$, but in addition, $\not{p}\psi = m\psi$, and the latter further restricts the spinor. Here we have lost the latter condition and must replace it by another. We choose the criterion that the spinors for each quark are pure quark states in the rest system of the excited state. Thus we add the three restrictions

$$P_\mu\gamma_{a\mu}h_i = m_ih_i,$$
$$P_\mu\gamma_{b\mu}h_i = m_ih_i, \qquad (21)$$
$$P_\mu\gamma_{c\mu}h_i = m_ih_i,$$

on the eigenstate h_i of the mass m_i, where P_μ is the four-momentum of the state and $P^2 = m_i^2$.

This again restricts the states, plays havoc with the sum rules, and will require attention in determining the adjustment factor. The way we handled this is discussed below.

The spinors should be normalized so that for each one, $\bar{u}u = 1$. Usually, in electrodynamics we normalize so that $\bar{u}u = 2m$ and say that the current matrix element is $\bar{u}_2\gamma_\mu u_1$. But now, since we are using for the current operator the matrix element $\bar{u}_2(\not{p}_2\gamma_\mu + \gamma_\mu\not{p}_1)u_1$, we shall have to normalize $\bar{u}u$ to 1 to get the same answer. (That is because we are using operators perturbing m^2, whereas in the usual Dirac theory the perturbations are of the operator $m = \not{p} - \not{A}$.)

MESON DYNAMICS

We represent mesons as states of a system of a quark and an antiquark. If p_a is the momentum of the quark and p_b the momentum of the antiquark, we may write for K

$$K = 2(p_a{}^2 + p_b{}^2) + \tfrac{1}{16}\Omega^2(u_a - u_b)^2, \qquad (22)$$

omitting a constant term. We choose the $\frac{1}{16}\Omega^2$ so that the Regge slope will be the same as for baryons, but we have, in our model, no explanation of this remarkable fact. Now we take $P = p_a + p_b$ and ζ for the internal momentum,

$$p_a = \tfrac{1}{2}P - \frac{1}{2\sqrt{2}}\zeta,$$
$$p_b = \tfrac{1}{2}P + \frac{1}{2\sqrt{2}}\zeta, \qquad (23)$$
$$z = -i\frac{\partial}{\partial\zeta} = \frac{(u_b - u_a)}{2\sqrt{2}},$$

to get

$$K = P^2 - \mathfrak{N},$$

where

$$-\mathfrak{N} = +\tfrac{1}{2}\zeta^2 + \tfrac{1}{2}\Omega^2z^2 = -\Omega c^*\cdot c \qquad (24)$$

and

$$\zeta = \left(\frac{\Omega}{2}\right)^{1/2}(c^* + c), \quad z = -i\left(\frac{1}{2\Omega}\right)^{1/2}(c^* - c).$$

The ground-state function h_0 (which would, in momentum space, be $\exp(\zeta^2/2\Omega) = \exp[(p_a - p_b)^2/\Omega]$) satisfies $c_\mu h_0 = 0$. Again we suppose that excited states are spacelike,

$$(P\cdot c)h = 0.$$

For the problems we discuss here, it is easiest to describe the antiquark simply as another quark of opposite charge. Then our vector current becomes

$$j_\mu^V = 2\sum_\alpha e_\alpha(\not{p}_\alpha\gamma_\mu + \gamma_\mu\not{p}_\alpha), \qquad (25)$$

with e_a equal to the charge of the quark and e_b the charge on the antiquark (e.g., $-\frac{2}{3}$ for $\bar{u}$, $+\frac{1}{3}$ for $\bar{s}$ and $\bar{d}$). Similar expressions obtain for the axial-vector current and its divergence, with the factor 3 replaced by 2 in (19) and (20).

Spinors are again projected (thinking of the antiquark as simply a quark of negative charge):

$$P_\mu\gamma_{a\mu}h_i = m_ih_i\,,$$
$$P_\mu\gamma_{b\mu}h_i = m_ih_i\,. \qquad (26)$$

ELECTROMAGNETIC INTERACTIONS

Having the states as described in the Appendix, we now turn to computing the matrix elements of various operators. We begin with the photoelectric matrix element. If the polarization of the outgoing photon is e_μ and it carries a momentum q from the initial state of four-momentum P_1 to the final state of momentum

$$P_2 = P_1 - q, \qquad (27)$$

then our mass-squared perturbing operator is $j_\mu^V e_\mu$, where j_μ^V is given in (18). Because the states are symmetrical in the quarks a, b, and c, each term in the sum on α gives the same result, so we can just take the term on the first quark a and multiply by 3. We move the operator $e^{iq\cdot u_a}$ to the left, so that $\not{p}_a e^{iq\cdot u_a} = e^{iq\cdot u_a}(\not{p}_a - \not{q})$. We use the Dirac matrices

$$\gamma_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \gamma_t \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$
$$i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \vec{\sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{28}$$

where $\vec{\sigma}$ are Pauli matrices, and get, putting $p_a = (\epsilon_a, \vec{p}_a)$, $q = (\nu, \vec{Q})$,

$$\mathfrak{M}^V = 9e_a e^{iq\cdot u_a}[(2\epsilon_a - \nu)e_t - \vec{\alpha}_a \cdot (e_t\vec{Q} - \nu\vec{e}) - (2\vec{p}_a - \vec{Q})\cdot\vec{e} + i\vec{\sigma}_a\cdot(\vec{Q}\times\vec{e})]. \tag{29}$$

The Dirac spinors for the initial state 1, and the final state 2, are taken in the direction of $P_1 = (E_1, \vec{P}_1)$ and $P_2 = (E_2, \vec{P}_2) = (E_1 - \nu, \vec{P}_1 - \vec{Q})$, respectively. Thus, in the representation (28),

$$\chi_1 = \left(\frac{E_1 + m_1}{2m_1}\right)^{1/2} \begin{pmatrix} \lambda_1 \\ \dfrac{\vec{\sigma}\cdot\vec{P}_1}{E_1 + m_1}\lambda_1 \end{pmatrix}$$

and

$$\bar{\chi}_2 = \left(\frac{E_2 + m_2}{2m_2}\right)^{1/2}\left(\lambda_2^*, -\lambda_2^*\frac{\vec{\sigma}\cdot\vec{P}_2}{E_2 + m_2}\right),$$

where λ_1 and λ_2 are two-component spinors describing the spins in the rest system of particles 1 and 2. Each one of the three quarks has a spinor of this kind with various λ corresponding to the spin. In the operator (29), although the dependence on a quark a is explicit, there is an implicit operator 1 on the spinors of quarks b and c.

Now we specialize to a particular coordinate system in which state 1 is at rest, $\vec{P}_1 = 0$, and the photon goes off in the z direction; $\vec{Q}$ is purely in the z direction. Note that

$$\vec{P}_2 = -\vec{Q} \text{ and } E_2 = (m_1^2 + m_2^2 - m_3^2)/2m_1,$$

where we have written m_3^2 for q^2 (for photons $m_3^2 = 0$). Note also that $\lambda_2 = \binom{1}{0}$, $\binom{0}{1}$ give directly the helicity amplitudes.

Now, $\bar{\chi}_2 = (\lambda_2^*, \lambda_2^*(\vec{\sigma}\cdot\vec{Q})/(2m_2 g^2))g$, where

$$g = \left(\frac{(m_1 + m_2)^2 - m_3^2}{4m_1 m_2}\right)^{1/2}. \tag{30}$$

The factor 1 on quark b now gives a factor like

$$(\bar{\chi}_{2b}\chi_{1b}) = g(\lambda_{2b}^*\lambda_{1b}), \tag{31}$$

or simply a factor g times the unit operator on the Pauli spinors. Quark c gives the same.

The expression (29) can now be evaluated as an operator on the Pauli spinors,

$$9g^3 e_a e^{iq\cdot u_a}\left[\left(2\epsilon_a - \nu - \frac{Q^2}{2m_2 g^2}\right)e_t - 2\vec{p}_a\cdot\vec{e} + [\vec{Q}\cdot\vec{e} + i\vec{\sigma}_a\cdot(\vec{Q}\times\vec{e})]\left(1 + \frac{\nu}{2m_2 g^2}\right)\right], \tag{32}$$

where ϵ_a and $\vec{p}_a$ are the time and space components of p_a.

Expressed in the normal-mode coordinates (4a), $p_a = \frac{1}{3}P - \frac{1}{3}\xi$ or $\epsilon_a = \frac{1}{3}m_1 - \frac{1}{3}\xi_t$, $\vec{p}_a = -\frac{1}{3}\vec{\xi}$. These and u_a can be further reduced to expressions in a and a^* via (12a). When the center-of-mass motion is taken out (a factor $\exp i[q\cdot\frac{1}{3}(u_a + u_b + u_c)]$), $e^{iq\cdot u_a}$ becomes

$$e^{-2iq\cdot x} = \exp\left[-\left(\frac{2}{\Omega}\right)^{1/2}(a^* - a)\cdot q\right] = \exp(q^2/\Omega)\exp\left[-\left(\frac{2}{\Omega}\right)^{1/2}(a^*\cdot q)\right]\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2}(q\cdot a)\right], \tag{33}$$

where we have used the commutation relations to reduce the expression to one in which all the a appear first on the initial state and all the a^* last, or directly on the final state. This makes the evaluation of matrix elements easy. In a like manner, we find

$$e^{-2iq\cdot x}\xi = \exp(q^2/\Omega)\exp\left[-\left(\frac{2}{\Omega}\right)^{1/2}(a^*\cdot q)\right]\left[\left(\frac{\Omega}{2}\right)^{1/2}a^* + \left(\frac{\Omega}{2}\right)^{1/2}(a - q)\right]\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2}(q\cdot a)\right], \tag{34}$$

so that

$$\mathfrak{M}^V = 9Fe_a\exp\left[-\left(\frac{2}{\Omega}\right)^{1/2}q\cdot a^*\right]\left\{\left[\tfrac{2}{3}m_1 - \tfrac{1}{3}\nu - \frac{Q^2}{2m_2 g^2} - \tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(a_t^* + a_t)\right]e_t + \tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(\vec{a}^* + \vec{a})\cdot\vec{e} + (\vec{Q}\cdot\vec{e})\left(\tfrac{1}{3} + \frac{\nu}{2m_2 g^2}\right) + i\vec{\sigma}_a\cdot(\vec{Q}\times\vec{e})\left(1 + \frac{\nu}{2m_2 g^2}\right)\right\}\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2}q\cdot a\right], \tag{35}$$

where we have replaced a factor $\exp(q^2/\Omega)\times g^3$ by F [defined in Eq. (15)]. We have already explained our reason for the replacement in the case of the factor $\exp(q^2/\Omega)$. The reason for including the g^3 is similar. The factor g in Eq. (31) seems surprising at first. With no direct perturbation on the b quark there should be no projection factor into a new direction, surely not one greater than unity (g is greater than unity). It comes because we artificially (i.e., not for dynamical reasons) take the Dirac spinor as that of a quark, and not of an antiquark, in the direction of the four-momentum of each baryon. Thus, when going from one direction P_1 to another P_2, the proportion of small components changes and there is a projection involved. If we took all possible states, quark and antiquark, for the spinor, a sum rule like (14) would remain valid, but taking only quark states gives too high a result. We have chosen to omit a factor g^3 for baryons (one g for each quark, hence g^2 in the case of mesons), and imagine it to be replaced, along with $\exp(q^2/\Omega)$, by our adjustment factor F.

Thus, explicitly, we replace a factor

$$g^3\exp(q^2/\Omega) \tag{36a}$$

by F in all baryon-baryon matrix elements, and

$$g^2\exp(q^2/2\Omega) \tag{36b}$$

by F in all meson-meson matrix elements, where F is chosen as in Eq. (15).

There is one further detail. We should like to describe the internal harmonic-oscillator excitation of states in their own rest system of coordinates, so that the algebra of the a^* is simple for each state. Hence in (35) we must consider a^* components t', x', y', z' in the system of the final states. This is easy to read from (35) because a^* appears only in four-vector combinations like $q\cdot a^* = q_\mu a_\mu^*$ or $e_\mu a_\mu^*$.

We have left the result (35) in general form so that we might someday use it for electron photoproduction, where $q^2\neq 0$ and $e_t\neq 0$, but we shall not make that application here. We may, however, stop to check the gauge invariance. From Eq. (3) and the manner in which we defined j_μ^V, it is easily proved, of course, if C is a true constant. Putting $(e_t, \vec{e}) = (\nu, \vec{Q})$ should give zero. The matrix we get with this substitution is

$$\mathfrak{M}^V_{\text{gauge}} = 9Fe_a \exp\left[-\left(\frac{2}{\Omega}\right)^{1/2} q\cdot a^*\right]\left[\tfrac{1}{3}(m_1^{\,2}-m_2^{\,2}) - \tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(a^*\cdot q + a\cdot q)\right]\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2} q\cdot a\right].$$

This makes transitions of internal motion having components excited by the operator $q\cdot a^*$. Suppose we consider first the oscillator in the q direction going from state n to state m, so the initial state is

$$\frac{1}{\sqrt{n!}\,|q|^n}(a^*\cdot q)^n|h_0\rangle,$$

and similarly for the excited state. We find

$$\begin{aligned}\langle m|\mathfrak{M}^V_{\text{gauge}}|n\rangle &= \text{factors}\,\langle h_0|(q\cdot a)^m \exp\left[-\left(\frac{2}{\Omega}\right)^{1/2} q\cdot a^*\right] \\ &\quad\times[m_1^{\,2}-m_2^{\,2}-\sqrt{2\Omega}(q\cdot a^* + q\cdot a)] \\ &\quad\times\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2} q\cdot a\right](q\cdot a^*)^n|h_0\rangle \\ &= \text{factors}[m_1^{\,2}-m_2^{\,2}-(n-m)\Omega].\end{aligned} \tag{37}$$

Thus, if the mass laws were exactly given by the harmonic motion, $m_2^{\,2}-m_1^{\,2}=(m-n)\Omega$, the result is zero. Because of the contributions in Eqs. (1) and (2) to the constant C, this is not strictly true. The operator e_a makes no change in strangeness nor in the spin state (doublet to quartet), so the dependence of C on the number of strange quarks or on the spin relation has no effect. But among doublet states we have mass differences, for example, we have 0.40 GeV^2 for $^2(\underline{10})$–$^2(\underline{8})$ of the $\underline{70}$ and 0.45 GeV^2 for $^2(\underline{8})$–$^2(\underline{1})$ [from the figure 1.25 GeV^2 in (1) for $^2(\underline{1})$ you must subtract 0.40 GeV^2 for the strangeness to compare with the Λ in $^2(\underline{8})$], and finally 0.42 GeV^2 for $^2(\underline{8})$ $\underline{56}$ to $^2(\underline{8})$ $\underline{70}$. To represent these by operators in C, a unitary spin exchange permutation operator would be necessary. (Very roughly, C seems to have contributions -0.40 times the mean of the spin-exchange operator, -0.40 times mean unitary-spin exchange, and $+0.40$ times mean space exchange; this is -0.53 for each pair interaction antisymmetric in both spin and unitary spin.) But such a term produces exchange currents, as is well known from the corresponding problem in nuclear physics. Thus our current is not strictly gauge-invariant if we use the true experimental masses in expressions for the matrix element. Nevertheless, we shall do just that and leave the omission of exchange currents as one of the complicating problems for the future.

In application to finding the photoelectric matrix elements needed for analysis of the production of π's by photons on protons or neutrons (the only cases for which data are available), we can make a number of simplifications in (35).

We take $q^2=0$, $q\cdot e=0$, $\vec{Q}$ in the z direction, and suppose the polarization vector $\vec{e}$ has transverse x and y components only; then, noting that we take the timelike oscillator of the final state in its ground state,

$$\mathfrak{M}^V = 9Fe_a \exp\left[+\left(\frac{2}{\Omega}\right)^{1/2} Qa_z^*\right]\left\{\tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(\vec{a}^* + \vec{a})\cdot\vec{e} + i\vec{\sigma}_a\cdot(\vec{Q}\times\vec{e})\left(1+\frac{\nu}{2m_2 g^2}\right)\right\}\exp\left[-\left(\frac{2}{\Omega}\right)^{1/2} Qa_z\right]. \tag{38}$$

The first term is from orbital current, the second from the Dirac magnetic moment of the quarks. Note that for $q^2=0$, $1+\nu/2m_2g^2 = 2m_1/(m_1+m_2)$.

In our applications here, the final state is always a nucleon with the oscillators in the ground state, so that all a^* in (38) may be replaced by zero. The resulting photoelectric amplitudes are given in Table I.

The first column gives the state the conventional symbol for its wave in pion scattering, and the second column its multiplet designation. We can consider the photoelectric amplitude as the amplitude for photoelectric disintegration into nucleon and γ, such that one state emits a photon of helicity +1 from either a z component of spin $+\frac{3}{2}$ (so the emitted nucleon has spin $+\frac{1}{2}$) or $+\frac{1}{2}$ (so the nucleon has spin $-\frac{1}{2}$). In columns 3 and 4 we give this helicity, and whether we are dealing with protons (p) or neutrons (n). In column 5 we give the formula for the matrix element of $\mathfrak{M}^V/F$ in each case, where

$$\lambda = \left(\frac{2}{\Omega}\right)^{1/2} Q,$$
$$Q = (m_1^2 - m_2^2)/2m_1, \tag{39}$$
$$\rho = \sqrt{\Omega}\,\frac{2m_1}{m_1+m_2}\,\lambda = \sqrt{2}\,(m_1 - m_2).$$

From $\langle f|\mathfrak{M}^V|i\rangle$ we have calculated the matrix element A that Walker[6] tabulates, which does not use the relativistic normalization, so that

$$\begin{aligned} A &= (4\pi e^2)^{1/2}(2m_1\times 2E_2\times 2Q)^{-1/2}\langle f|\mathfrak{M}^V|i\rangle \\ &= (2\pi e^2)^{1/2}[m_1/(m_1^4 - m_2^4)]^{1/2}\langle f|\mathfrak{M}^V|i\rangle . \end{aligned} \tag{40}$$

First we give our value for A, then Walker's value computed from a nonrelativistic quark model in which he chose the quark mass equal to $\frac{1}{3}$ the proton mass, and the quark moment equal to the corresponding Dirac moment. We have no choice of parameters here at all, except in our form factor F. Finally, in the last column, we give experimental figures, without probable errors, from Walker.

TABLE I. Photoelectric matrix elements.

State	Multiplet	J_z	I_z	$\langle f\|\mathfrak{M}^V\|i\rangle/F$	A (GeV$^{-1/2}$)	$A^{\rm NR}$ (GeV$^{-1/2}$)	$A^{\rm exp}$ (GeV$^{-1/2}$)
$P_{33}(1236)$	${}^4\underline{10}_{3/2}[\underline{56},0^+]_0$	$+\frac{3}{2}$	p	$-\sqrt{6}\rho$	−0.187	−0.178	−0.244
		$+\frac{1}{2}$	p	$-\sqrt{2}\rho$	−0.108	−0.103	−0.138
$D_{13}(1520)$	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$+\frac{3}{2}$	p	$+\sqrt{\Omega}$	+0.109	+0.112	+0.151
		$+\frac{1}{2}$	p	$-\sqrt{3}\lambda\rho+(\frac{1}{3})^{1/2}\sqrt{\Omega}$	−0.034	−0.029	−0.026
		$+\frac{3}{2}$	n	$-\sqrt{\Omega}$	−0.109	−0.112	−0.132
		$+\frac{1}{2}$	n	$+(\frac{1}{3})^{1/2}\lambda\rho-(\frac{1}{3})^{1/2}\sqrt{\Omega}$	−0.031	−0.030	
$S_{11}(1535)$	${}^2\underline{8}_{1/2}[\underline{70},1^-]_1$	$+\frac{1}{2}$	p	$+(\frac{3}{2})^{1/2}\lambda\rho+(\frac{2}{3})^{1/2}\sqrt{\Omega}$	+0.156	+0.160	+0.096
		$+\frac{1}{2}$	n	$-(\frac{1}{6})^{1/2}\lambda\rho-(\frac{2}{3})^{1/2}\sqrt{\Omega}$	−0.108	−0.109	−0.118
$D_{15}(1670)$	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$+\frac{3}{2}$	p	0	0	0	0.040?
		$+\frac{1}{2}$	p	0	0	0	~0
		$+\frac{3}{2}$	n	$-(\frac{3}{5})^{1/2}\lambda\rho$	−0.053	−0.053	
		$+\frac{1}{2}$	n	$+(\frac{3}{10})^{1/2}\lambda\rho$	+0.038	+0.038	
$S_{31}(1650)$	${}^2\underline{10}_{1/2}[\underline{70},1^-]_1$	$+\frac{1}{2}$	p	$-(\frac{1}{6})^{1/2}\lambda\rho+(\frac{2}{3})^{1/2}\sqrt{\Omega}$	+0.047	+0.047	
$D_{33}(1670)$	${}^2\underline{10}_{3/2}[\underline{70},1^-]_1$	$+\frac{3}{2}$	p	$+\sqrt{\Omega}$	+0.084	+0.091	
		$+\frac{1}{2}$	p	$+(\frac{1}{3})^{1/2}\lambda\rho+(\frac{1}{3})^{1/2}\sqrt{\Omega}$	+0.088	+0.092	
$P_{11}(1470)$	${}^2\underline{8}_{1/2}[\underline{56},0^+]_2$	$+\frac{1}{2}$	p	$[+(\frac{3}{4})^{1/2}\lambda\rho]\lambda$	+0.027	+0.032	
		$+\frac{1}{2}$	n	$[-(\frac{1}{3})^{1/2}\lambda\rho]\lambda$	−0.018	−0.020	
$F_{15}(1688)$	${}^2\underline{8}_{5/2}[\underline{56},2^+]_2$	$+\frac{3}{2}$	p	$[+(\frac{4}{5})^{1/2}\sqrt{\Omega}]\lambda$	+0.059	+0.070	+0.139
		$+\frac{1}{2}$	p	$[-(\frac{9}{10})^{1/2}\lambda\rho+(\frac{2}{5})^{1/2}\sqrt{\Omega}]\lambda$	−0.010	−0.015	~0
		$+\frac{3}{2}$	n	0	0	0	~0
		$+\frac{1}{2}$	n	$[+(\frac{2}{5})^{1/2}\lambda\rho]\lambda$	+0.035	+0.041	
$\omega(784)$	${}^3S_1$	0	π	$(\frac{1}{2})^{1/2}\rho$	0.172		0.13 ± 0.01
$\Lambda(1520)$	${}^2\underline{1}_{3/2}[\underline{70},1^-]_1$	$+\frac{3}{2}$	Λ	$+\frac{5}{6}\sqrt{\Omega}$	+0.107		0.095 ± 0.010
		$+\frac{1}{2}$	Λ	$-\frac{5}{6}[\sqrt{3}\lambda\rho-(\frac{1}{3})^{1/2}\sqrt{\Omega}]$	+0.012		

TABLE II. Transition amplitudes for decays into ground-state baryons and pseudoscalar mesons.

State (mass)	$(\text{Mass})^2$	Multiplet	Mode	$\langle f \mid \mathfrak{M}^P \mid i \rangle / F$
$P_{33}(1236)$	1.53	${}^4\underline{10}_{3/2}[\underline{56},0^+]_0$	$N\pi$	$+\frac{8}{3}\sqrt{2}\gamma$
$S_{01}(1405)$	1.97	${}^2\underline{1}_{1/2}[\underline{70},1^-]_1$	$\Sigma\pi$	$+\sqrt{2}[\lambda\gamma(1+\delta)-3\beta]$
$D_{03}(1520)$	2.30	${}^2\underline{1}_{3/2}[\underline{70},1^-]_1$	$\Sigma\pi$	$-2[\lambda\gamma(1+\delta)]$
$S_{11}(1535)$	2.36	${}^2\underline{8}_{1/2}[\underline{70},1^-]_1$	$N\pi$	$+\frac{4}{9}\sqrt{6}[\lambda\gamma(1+\delta)-3\beta]$
$D_{13}(1520)$	2.31	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$N\pi$	$-\frac{8}{9}\sqrt{3}[\lambda\gamma(1+\delta)]$
$S_{11}(1700)$	2.89	${}^4\underline{8}_{1/2}[\underline{70},1^-]_1$	$N\pi$	$+\frac{2}{9}\sqrt{6}[\lambda\gamma(1+\delta)-3\beta]$
$D_{13}(\quad)$		${}^4\underline{8}_{3/2}[\underline{70},1^-]_1$	$N\pi$	$-\frac{2}{45}\sqrt{30}[\lambda\gamma(1+\delta)]$
$D_{15}(1670)$	2.79	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$N\pi$	$-\frac{2}{15}\sqrt{30}[\lambda\gamma(1+\delta]$
$S_{31}(1650)$	2.72	${}^2\underline{10}_{1/2}[\underline{70},1^-]_1$	$N\pi$	$+\frac{2}{9}\sqrt{6}[\lambda\gamma(1+\delta)-3\beta]$
$D_{33}(1670)$	2.79	${}^2\underline{10}_{3/2}[\underline{70},1^-]_1$	$N\pi$	$-\frac{4}{9}\sqrt{3}[\lambda\gamma(1+\delta)]$
$P_{11}(1470)$	2.16	${}^2\underline{8}_{1/2}[\underline{56},0^+]_2$	$N\pi$	$-\frac{5}{9}\sqrt{3}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{33}(\quad)$		${}^4\underline{10}_{3/2}[\underline{56},0^+]_2$	$N\pi$	$-\frac{4}{9}\sqrt{6}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{13}(\quad)$		${}^2\underline{8}_{3/2}[\underline{56},2^+]_2$	$N\pi$	$+\frac{2}{9}\sqrt{15}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{15}(1688)$	2.85	${}^2\underline{8}_{5/2}[\underline{56},2^+]_2$	$N\pi$	$+\frac{1}{3}\sqrt{10}[\lambda\gamma(1+2\delta)]\lambda$
$P_{31}(1910)$	3.65	${}^4\underline{10}_{1/2}[\underline{56},2^+]_2$	$N\pi$	$+\frac{8}{45}\sqrt{15}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$P_{33}(\quad)$		${}^4\underline{10}_{3/2}[\underline{56},2^+]_2$	$N\pi$	$-\frac{8}{45}\sqrt{15}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{35}(1890)$	3.57	${}^4\underline{10}_{5/2}[\underline{56},2^+]_2$	$N\pi$	$-\frac{8}{105}\sqrt{35}[\lambda\gamma(1+2\delta)]\lambda$
$F_{37}(1950)$	3.80	${}^4\underline{10}_{7/2}[\underline{56},2^+]_2$	$N\pi$	$+\frac{8}{105}\sqrt{210}[\lambda\gamma(1+2\delta)]\lambda$
$P_{01}(\quad)$		${}^2\underline{1}_{1/2}[\underline{70},0^+]_2$	$\Sigma\pi$	$+\frac{1}{2}\sqrt{2}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{11}(1780)$	3.17	${}^2\underline{8}_{1/2}[\underline{70},0^+]_2$	$N\pi$	$+\frac{2}{9}\sqrt{6}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{13}(\quad)$		${}^4\underline{8}_{3/2}[\underline{70},0^+]_2$	$N\pi$	$+\frac{1}{9}\sqrt{6}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{31}(\quad)$		${}^2\underline{10}_{1/2}[\underline{70},0^+]_2$	$N\pi$	$+\frac{1}{9}\sqrt{6}[\lambda\gamma(1+2\delta)-2\beta]\lambda$
$P_{03}(\quad)$		${}^2\underline{1}_{3/2}[\underline{70},2^+]_2$	$\Sigma\pi$	$+\frac{1}{5}\sqrt{10}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{05}(\quad)$		${}^2\underline{1}_{5/2}[\underline{70},2^+]_2$	$\Sigma\pi$	$-\frac{1}{5}\sqrt{15}[\lambda\gamma(1+2\delta)]\lambda$
$P_{13}(1860)$	3.46	${}^2\underline{8}_{3/2}[\underline{70},2^+]_2$	$N\pi$	$+\frac{4}{45}\sqrt{30}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{15}(\quad)$		${}^2\underline{8}_{5/2}[\underline{70},2^+]_2$	$N\pi$	$+\frac{4}{15}\sqrt{5}[\lambda\gamma(1+2\delta)]\lambda$
$P_{11}(\quad)$		${}^4\underline{8}_{1/2}[\underline{70},2^+]_2$	$N\pi$	$-\frac{2}{45}\sqrt{15}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$P_{13}(\quad)$		${}^4\underline{8}_{3/2}[\underline{70},2^+]_2$	$N\pi$	$+\frac{2}{45}\sqrt{15}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{15}(\quad)$		${}^4\underline{8}_{5/2}[\underline{70},2^+]_2$	$N\pi$	$+\frac{2}{105}\sqrt{35}[\lambda\gamma(1+2\delta)]\lambda$
$F_{17}(1990)$	3.96	${}^4\underline{8}_{7/2}[\underline{70},2^+]_2$	$N\pi$	$-\frac{2}{105}\sqrt{210}[\lambda\gamma(1+2\delta)]\lambda$
$P_{33}(\quad)$		${}^2\underline{10}_{3/2}[\underline{70},2^+]_2$	$N\pi$	$+\frac{2}{45}\sqrt{30}[\lambda\gamma(1+2\delta)-5\beta]\lambda$
$F_{35}(\quad)$		${}^2\underline{10}_{5/2}[\underline{70},2^+]_2$	$N\pi$	$-\frac{2}{15}\sqrt{5}[\lambda\gamma(1+2\delta)]\lambda$
$\Delta(2420)$	5.86	${}^4\underline{10}_{11/2}[\underline{56},4^+]_4$	$N\pi$	$+\frac{4}{3}(\frac{1}{77})^{1/2}[\lambda\gamma(1+4\delta)]\lambda^3$

PSEUDOSCALAR-MESON INTERACTIONS

We shall calculate decay widths of excited hadrons into a pseudoscalar meson and another hadron by replacing the pseudoscalar-meson interaction by the divergence of an axial-vector current (20), in the way usually done for PCAC (partial conservation of axial-vector current) or in studying the Goldberger-Treiman relation. The "charges" e'_α are replaced by the SU_3, 3×3 matrix λ appropriate for the particular member of the meson nonet. For neutral mesons, where λ is diagonal, the formula (20) is directly applicable, the "charges" e'_a being the diagonal elements of λ. Details are given in the Appendix.

Proceeding in exactly the same way with the axial-vector current (19) as with the vector current, calling the polarization e'_μ and the charge e'_a, we find, for the expression analogous to (35),

$$\begin{aligned}\mathfrak{M}^A = 9iFe'_a \exp\left[-\left(\frac{2}{\Omega}\right)^{1/2}(q\cdot a^*)\right]\Bigg\{&(\vec{\sigma}_a\cdot\vec{Q})\left[\tfrac{1}{3}e'_t+\frac{\nu e'_t}{2m_2g^2}-\frac{2}{3}\frac{\vec{Q}\cdot\vec{e}\,'}{2m_2g^2}\right]\\&+(\vec{\sigma}_a\cdot\vec{e}\,')\left[\tfrac{1}{3}(m_1+m_2)-\tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(a_t+a_t^*)-\tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}\frac{\vec{Q}\cdot(\vec{a}+\vec{a}^*)}{2m_2g^2}\right]\\&+\left(\frac{\Omega}{2}\right)^{1/2}\vec{\sigma}_a\cdot(\vec{a}+\vec{a}^*)\left(\tfrac{2}{3}e'_t+\tfrac{2}{3}\frac{\vec{Q}\cdot\vec{e}\,'}{2m_2g^2}\right)+i\,\tfrac{2}{3}\left(\frac{\Omega}{2}\right)^{1/2}(\vec{a}+\vec{a}^*)\cdot\frac{\vec{Q}\times\vec{e}\,'}{2m_2g^2}\Bigg\}\exp\left[+\left(\frac{2}{\Omega}\right)^{1/2}(q\cdot a)\right]. \end{aligned}\tag{41}$$

Putting $e'_\mu=-iq_\mu$, we find the matrix element of the divergence of the axial-vector current which we shall use as the pseudoscalar-meson coupling; so, calling the matrix $\mathfrak{M}^P$, we find from (41), taking $\vec{Q}$ in the z direction, $q=(\nu,0,0,Q)$,

$$\begin{aligned}\mathfrak{M}^P = 6Fe'_a e^{+\lambda a_z^*}[&\gamma\sigma_{az}-\gamma\lambda\delta\sigma_{az}(a_z^*+a_z)\\&+\beta\vec{\sigma}_a\cdot(\vec{a}^*+\vec{a})]\,e^{-\lambda a_z},\end{aligned}\tag{42}$$

where (we have assumed that a_t and a_t^* give zero on our states)

$$\begin{aligned}\lambda&=\left(\frac{2}{\Omega}\right)^{1/2}Q,\\ \gamma&=m_1Q\left(1+\frac{3m_3^{\,2}}{(m_1+m_2)^2-m_3^{\,2}}\right),\\ \beta&=\left(\frac{\Omega}{2}\right)^{1/2}(m_1-m_2),\\ \gamma\delta&=m_1Q\,\frac{\Omega}{(m_1+m_2)^2-m_3^{\,2}}\,.\end{aligned}\tag{43}$$

For calculational use, we write

$$\vec{\sigma}\cdot(\vec{a}^*+\vec{a})=\sigma_z(a_z^*+a_z)+\sqrt{2}\,\sigma_+(a_-^*-a_+)-\sqrt{2}\,\sigma_-(a_+^*-a_-),\tag{44}$$

where

$$\sigma_\pm=\tfrac{1}{2}(\sigma_x\pm i\sigma_y),$$

$$a_\pm=\mp\frac{1}{\sqrt{2}}(a_x\mp ia_y).$$

These matrices are now evaluated in a direct way from the wave functions in the manner described in the Appendix. The results of these calculations for pseudoscalar-meson emission amplitudes appear in Table II, where we give one representative amplitude for each multiplet. Amplitudes for other states in the same multiplet can be found from the F/D values in Table VIII in the Appendix.

The calculations with meson states instead of baryons proceed along the same lines in an obvious way but are simpler, so no details will be given. Because the states are not fully symmetric in the quark and antiquark, one does not avoid the sum over α, but computes each term (or else one can divide the matrix into a symmetric or antisym-

TABLE III. Transition helicity amplitudes for meson decays.

State (mass)	Multiplet	Mode	A_0/F	A_1/F
$\rho^+(765)$	$^3S_1$	$\pi^+\pi^0$	4γ	0
$B^+(1235)$	$^1P_1$	$\omega^0\pi^+$	$+2\sqrt{2}[\gamma\lambda(1+\delta)-\beta]$	$-2\sqrt{2}\beta$
$\delta^+(966)$?	$^3P_0$	$\eta^0\pi^+$	$+\frac{2}{3}\sqrt{2}[\gamma\lambda(1+\delta)-3\beta]$	0
$A_1^+(1070)$	$^3P_1$	$\rho^+\pi^0$	-4β	$+2[\gamma\lambda(1+\delta)-2\beta]$
$A_2^+(1300)$	$^3P_2$	$\rho^+\pi^0$	0	$+2[\gamma\lambda(1+\delta)]$

TABLE IV. Transition rates for decays into ground-state baryons and pseudoscalar mesons.

State	Multiplet	Mode	Γ_{calc} (MeV)	Γ_{exp} (MeV)	$\ln(\Gamma_{\text{calc}}/\Gamma_{\text{exp}})$
$\Delta(1236)$	${}^4\underline{10}_{3/2}[\underline{56},0^+]_0$	$N\pi$	94	120	−0.2
$\Sigma(1385)$	${}^4\underline{10}_{3/2}[\underline{56},0^+]_0$	$\Lambda\pi$	35	32	+0.1
		$\Sigma\pi$	4	4	0
$\Xi(1530)$	${}^4\underline{10}_{3/2}[\underline{56},0^+]_0$	$\Xi\pi$	12	7	+0.5
$\Lambda(1405)$	${}^2\underline{1}_{1/2}[\underline{70},1^-]_1$	$\Sigma\pi$	56	40	+0.3
$\Lambda(1520)$	${}^2\underline{1}_{3/2}[\underline{70},1^-]_1$	$N\bar{K}$	7	7	0
		$\Sigma\pi$	12	7	+0.5
$N(1535)$	${}^2\underline{8}_{1/2}[\underline{70},1^-]_1$	$N\pi$	220	40	+1.7
		$N\eta$	71	77	−0.1
$\Lambda(1670)$	${}^2\underline{8}_{1/2}[\underline{70},1^-]_1$	$\Sigma\pi$	22	11	+0.7
		$\Lambda\eta$	6	8	−0.3
		$N\bar{K}$	415	5	+4.4
$N(1520)$	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$N\pi$	105	60	+0.6
		$N\eta$	0.2	~0.7	~−1.3
$\Sigma(1670)$	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$N\bar{K}$	3		
		$\Lambda\pi$	6		
		$\Sigma\pi$	49		
$\Lambda(1690)$	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$N\bar{K}$	102	16	+1.9
		$\Sigma\pi$	11	21	−0.6
$\Xi(1820)$	${}^2\underline{8}_{3/2}[\underline{70},1^-]_1$	$\Lambda\bar{K}$	15		
		$\Xi\pi$	4		
		$\Sigma\bar{K}$	17		
		$\Xi^*\pi$	10		
$N(1700)$	${}^4\underline{8}_{1/2}[\underline{70},1^-]_1$	$N\pi$	45	182	−1.4
		$N\eta$	112		
		ΛK	0 (74)	13	
$N(1670)$	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$N\pi$	36	60	−0.5
		$N\eta$	7	<1	
		ΛK	0 (0.2)	<0.1	
$\Sigma(1765)$	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$N\bar{K}$	66	53	+0.2
		$\Lambda\pi$	25	17	+0.4
		$\Sigma\pi$	10	~1	
		$\Sigma^*\pi$	6	15	−0.9
$\Lambda(1830)$	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$N\bar{K}$	0 (48)	11	
		$\Sigma\pi$	73	33	+0.8
$\Xi(1930)$	${}^4\underline{8}_{5/2}[\underline{70},1^-]_1$	$\Xi\pi$	83		
		$\Lambda\bar{K}$	24		
$\Delta(1650)$	${}^2\underline{10}_{1/2}[\underline{70},1^-]_1$	$N\pi$	25	41	−0.5
$\Sigma(1750)$	${}^2\underline{10}_{1/2}[\underline{70},1^-]_1$	$N\bar{K}$	14	~10	~+0.3
		$\Lambda\pi$	9		
		$\Sigma\eta$	4		
$\Delta(1670)$	${}^2\underline{10}_{3/2}[\underline{70},1^-]_1$	$N\pi$	30	31	0
$N(1470)$	${}^2\underline{8}_{1/2}[\underline{56},0^+]_2$	$N\pi$	8	150	−2.9
$N(1688)$	${}^2\underline{8}_{5/2}[\underline{56},2^+]_2$	$N\pi$	64	75	−0.1
		$\Lambda\bar{K}$	0.07	<0.1	
		$N\eta$	0.27	<0.6	
$\Sigma(1915)$	${}^2\underline{8}_{5/2}[\underline{56},2^+]_2$	$N\bar{K}$	3	8	−1.0
		$\Lambda\pi$	15	5	+1.1
		$\Sigma\pi$	24	3	+2.1

TABLE IV. *(Continued)*

State	Multiplet	Mode	$\Gamma_{\rm calc}$ (MeV)	$\Gamma_{\rm exp}$ (MeV)	$\ln(\Gamma_{\rm calc}/\Gamma_{\rm exp})$
$\Lambda(1815)$	$^2\underline{8}_{5/2}[\underline{56}, 2^+]_2$	$N\bar{K}$	35	53	−0.4
		$\Sigma\pi$	13	9	+0.4
		$\Sigma^*\pi$	16	14	+0.1
$\Xi(1930)$	$^2\underline{8}_{5/2}[\underline{56}, 2^+]_2$	$\Xi\pi$	1		
		$\Lambda\bar{K}$	2		
		$\Sigma\bar{K}$	16		
$\Delta(1910)$	$^4\underline{10}_{1/2}[\underline{56}, 2^+]_2$	$N\pi$	151	82	+0.6
$\Delta(1890)$	$^4\underline{10}_{5/2}[\underline{56}, 2^+]_2$	$N\pi$	18	47	−1.0
$\Delta(1950)$	$^4\underline{10}_{7/2}[\underline{56}, 2^+]_2$	$N\pi$	103	90	+0.1
		ΣK	7	5	+0.3
		$\Delta\pi$	43	~100	
		Σ^*K	0.1	3	
$\Sigma(2030)$	$^4\underline{10}_{7/2}[\underline{56}, 2^+]_2$	$N\bar{K}$	28	27	0
		$\Lambda\pi$	37	35	0
		$\Sigma\pi$	17	3	+1.7
		ΞK	1	<2	
$N(1780)$	$^2\underline{8}_{1/2}[\underline{70}, 0^+]_2$	$N\pi$	0.5	120	−5.5
$N(1860)$	$^2\underline{8}_{3/2}[\underline{70}, 2^+]_2$	$N\pi$	75	90	−0.2
		ΛK	22	<53	
		$N\eta$	19		
$N(1990)$	$^4\underline{8}_{7/2}[\underline{70}, 2^+]_2$	$N\pi$	15	26	−0.5
$\Delta(2420)$	$^4\underline{10}_{11/2}[\underline{56}, 4^+]_4$	$N\pi$	25	34	−0.3

metric sum). All couplings are either pure F or pure D. The expression for $\mathfrak{M}^P$ is the same as (42) with a replaced by $c/\sqrt{2}$, a factor of 4 instead of 6 in front, and γ for mesons changed to

$$\gamma = m_1 Q\left(1 + \frac{2m_3^2}{(m_1+m_2)^2 - m_3^2}\right). \tag{45}$$

In this meson case it is a quantity $g^2\exp(q^2/2\Omega)$ multiplying all matrix elements that is replaced by F of expression (15). Representative amplitudes are given in Table III.

The widths were computed from the relativistic rate formula [derived from (10)]

$$\Gamma = \frac{Q}{8\pi m_1^2}\,\frac{2R}{2J+1}\,|\sqrt{4\pi}\,f_R\,\mathfrak{M}|^2\,. \tag{46}$$

Here $2R/(2J+1)$ is a multiplicity factor, and J is the initial-state angular momentum when the matrix element is for one helicity only. The factor 2 in $2R$ is from the two helicities of the final baryon when we consider decays into baryons with spin $\frac{1}{2}$. R is the inverse of the squared Clebsch-Gordan coefficient connecting the initial isospin state with the isospin of the final state. For instance, if we calculate the amplitude for the decay of an excited proton into a proton and π^0, then $R = (\sqrt{3})^2 = 3$. (This R depends on our choice of exactly which typical matrix element is calculated in Tables IV and V. It would not be necessary if we would calculate each real possibility and sum.) For decimet-octet transitions, R is given in the Appendix, Table VII. For octet-octet, $R = 3$ for $N \to N\pi$, $N \to \Sigma^0 K$, $\Lambda \to \Sigma\pi$, $\Xi \to \Xi\pi$, $\Xi \to \Sigma K$, while $R = 2$ for $\Lambda \to N\bar{K}$, $\Sigma^+ \to \Sigma\pi$, $\Sigma^0 \to N\bar{K}$, and $R = 1$ for the other cases (Λ or η^0 in the final state).

We have written the coupling as $\sqrt{4\pi}\,f_R$. We shall adjust this constant of proportionality so that the majority of rates are fitted. In our numerical work, we calculated using that value for the constant so that the predicted charged pion-nucleon coupling is exactly the one observed, $f^2 = 0.08$. This means we use $f_R = (0.08)^{1/2} \times \frac{3}{5}/m_\pi = 1.21\ \text{GeV}^{-1}$, leaving discussion of any further adjustment (of about 5%, as it turned out) to after other rates are worked out, so this is an adjustable constant of our theory.

The other constant, $\Omega = 1.05\ \text{GeV}^2$, we chose from the mass spectra and considered it not adjustable. The only other adjustable constant was in the exponent of our adjustment factor, in Eq. (15).

The results of the calculations of pseudoscalar-meson decay rates are given for hyperon decays

TABLE V. Meson classification and decay rates.

State(mass)	(Mass)^2	Multiplet	Mode	Γ_{calc} (MeV)	Γ_{exp} (MeV)	$\ln(\Gamma_{\text{calc}}/\Gamma_{\text{exp}})$
$\eta'(958)$	0.92					
$\eta(549)$	0.30	$^1S_0$				
$K(494)$	0.24					
$\pi(140)$	0.02					
$\phi(1019)$	1.04		$K\bar{K}$	10	3.2	+1.1
			$\rho\pi$	0	<1	
$\omega(784)$	0.61	$^3S_1$	$\pi\pi$	0		
$K^*(892)$	0.80		$K\pi$	66	51	+0.3
			πK	159	51	+1.1
$\rho(765)$	0.58		$\pi\pi$	157	125	+0.2
$K^*(1300)$	1.69					
$B(1235)$	1.53	$^1P_1$	$\omega\pi$	85	102	−0.2
$K^*(1240)$	1.54		$K^*\pi$	60	90	−0.4
$A_1(1070)$	1.14	$^3P_1$	$\rho\pi$	161	95	+0.5
$f'(1514)$	2.29		$K\bar{K}$	103	52	+0.7
			$\bar{K}K^*+K\bar{K}^*$	15	7	+0.8
			$\pi\pi$	0	<10	
$f(1260)$	1.59		$\pi\pi$	244	145	+0.5
			$K\bar{K}$	13	~5	+0.9
$K^*(1420)$	2.01		$K^*\pi$	22	35	−0.5
		$^3P_2$	ρK	8	8	0
			ωK	2	4	−0.7
			$K\pi$	87	48	+0.6
			πK	140	48	+1.1
			$K\eta$	5	~2	+0.9
			ηK	4	~2	+0.7
$A_2(1300)$	1.69		$\rho\pi$	66	64	+0.0
			$\eta\pi$	22	16	+0.3
			$\pi\eta$	45	16	+1.0
			$\bar{K}K$	17	10?	+0.5?

in Table IV. In this table, the first and second columns give the state and multiplet assignment, the third gives the mode of decay. We do not give the calculated matrix elements, but rather only the fully calculated width Γ to be compared with experiment. Next we give Γ_{exp} from the 1970 Particle Properties Tables, and to make comparisons the last column gives $\ln(\Gamma_{\text{calc}}/\Gamma_{\text{exp}})$ for those cases where $\Gamma_{\text{calc}} \neq 0$. In cases where the theoretical result is zero because the numerical coefficient in front of the amplitudes in Table II is zero for some special values of (F, D), we give the answer as $O(x)$, where x is the value one would get for Γ if the coefficient were unity. So working backward one can see how small the data indicate that the coefficient in fact is. We see that none of these predicted zeros presents any real problem; the experimental value is small enough for all of them.

The corresponding table for meson states is Table V. In this table we represent the A_2 as a single resonance of width 90 MeV.[7]

COMPARISON TO EXPERIMENT

1. Diagonal Matrix Elements

We discuss first the diagonal elements, beginning with the matrix elements of currents (35) for the fundamental baryon octet. Using $q \cdot e = 0$ or $\vec{Q} \cdot \vec{e} = \nu e_t$, we find that the current couples as

$$2mF\left[e_t\left(1+\frac{3q^2}{4m^2g^2}\right)(1,0)+i\vec{\sigma}\cdot\frac{(\vec{Q}\times\vec{e})}{2m}\times\left(1+\frac{\nu}{2mg^2}\right)3\left(\tfrac{2}{3},1\right)\right]. \tag{47}$$

Here $m_1 = m_2$, so that [see Eq. (30)]

$$g=(1-q^2/4m^2)^{1/2} \text{ and } 1+\nu/2mg^2=1/g^2,$$

with q^2 negative. The factor $2m$ appears because this is the operator for perturbation of m^2 and is $2m$ times the ordinary perturbation in energy at rest. The numbers in parentheses are the (F, D) to be taken in the U-spin direction: $F+\frac{1}{3}D$ for the proton, $-\frac{2}{3}D$ for the neutron.

Studying first the case $q^2 \to 0$, so the adjustment factor is $F=1$, we obtain the correct charge, of course, to multiply e_t and a magnetic moment [for $i(\vec{Q}\times\vec{e})$ is the magnetic field] of +3.00 for the proton and −2.00 for the neutron, in their own magnetons. These results are well known (if one assumes that the quark mass should be $\frac{1}{3}$ the particle mass and carry its own Dirac moment; but here that result is automatic: no choice of a parameter is involved); they compare very well with experiment (2.79 and −1.91). For the other members of the octet we get the usual SU_3 predictions if each is measured in its own magneton. Thus the Λ should have a magnetic moment $+\frac{1}{2}$ that of the neutron, or −1.00 of its magneton, or −0.84 nuclear magneton (the experimental result[3] is -0.73 ± 0.16).

If for finite negative q^2 we compare (47) to the usual expression in terms of the conventional form factors, which works out as

$$2mg^{-1}\left(e_t G_E(q^2)+i\vec{\sigma}\cdot\frac{(\vec{Q}\times\vec{e})}{2m}G_M(q^2)\right), \tag{48}$$

we see that we are predicting that

$$\mu G_E/G_M = 1+q^2/2m^2 \tag{49}$$

(μ is the magnetic moment). Experiment[8] indicates that this ratio does decrease with increasing $-q^2$, but slower than (49). For the magnetic form factor we get

$$\frac{G_M}{\mu}=g^{-1}F=\left(1-\frac{q^2}{4m^2}\right)^{-1/2}\exp\left[\frac{q^2}{2\Omega}\left(1-\frac{q^2}{4m^2}\right)\right], \tag{50}$$

with m^2 and Ω about 1 GeV^2; q^2 is negative.

The result (50) is completely wrong, because experiments indicate a function roughly like

$$G_M=(1-q^2/0.71)^{-2}, \tag{51}$$

which falls off faster at first [like $1+2.8q^2$ instead of like (50), which goes as $1+0.64q^2$] and is definitely slower than a Gaussian for large q^2. We may not expect good results from our model for large q^2, for it is based on getting low-energy resonances correctly, but the small-q^2 results show that something else must be seriously wrong. We have not represented the ρ or ω pole; but to start in that direction requires an elaboration of our naive model that we do not know yet how to define generally. Clearly the model is too simple. The replacement (36) is not the cause of the serious discrepancy between (50) and (51); if we had not made that replacement we would have found $G_M=(1-q^2/4m^2)\times\exp(q^2/\Omega)$, going as $1+0.72q^2$ for small q^2.

For the axial-vector current for a proton at rest, we obtain the well-known result that the axial-vector coupling should be

$$g_A=\tfrac{5}{3}=1.67, \tag{52}$$

while the experiment is nearer 1.22. This is not much more discrepancy than we will accept for later matrix-element comparisons. But a diagonal element for a low state should be particularly good. The predicted F/D ratio here is $\frac{2}{3}$. (A recent evaluation[9] of the leptonic hyperon decay gives $F=0.49$, $D=0.74$.) This is a specific but well-known prediction of the SU_6 aspect of the model.

We can calculate, from matrix elements of j_μ^V, the form factor for the coupling to K and π in the $K\pi e\nu$ or $K\pi\mu\nu$ decays. This is usually written

$$f_+(q^2)(K_\mu+\pi_\mu)+f_-(q^2)(K_\mu-\pi_\mu). \tag{53}$$

We find

$$F\left[\left(1+\frac{2q^2}{(m_K+m_\pi)^2-q^2}\right)(K_\mu+\pi_\mu) - \frac{2(m_K{}^2-m_\pi{}^2)}{(m_K+m_\pi)^2-q^2}(K_\mu-\pi_\mu)\right]. \tag{54}$$

This gives us a predicted value of

$$\xi=f_-(0)/f_+(0)=-1.11, \tag{55}$$

while polarization experiments[10] give -0.94 ± 0.20. If we write $f(q^2)$ as $f(0)(1+\lambda q^2/m_\pi{}^2)$, we predict [the factor F is $0.956\ (1+0.013q^2/m_\pi{}^2)$]

$$\lambda_+=0.11,\quad \lambda_-=0.06,$$

whereas, experimentally, $\lambda_+\approx0.04$. These λ's are rather subtle properties; the agreement with ξ is probably more significant. The experimental fact[11] that $f_+(0)=0.94\pm0.05$ agrees with our predicted 0.96. Had we used the factor $g^2\exp(q^2/2\Omega)$ instead of F here, we would have had a very serious disagreement, for $g^2=1.45$, so $f_+(0)$ would be 1.45. All theoretical arguments and experimental results confirm that there should not be such a large renormalization of vector current and that $f_+(0)$ should be very near 1.

The formula (54) predicts a ratio for $\Gamma(K_{\mu3})/\Gamma(K_{e3})$ of 0.65, whereas experiment[3] gives 0.65 ± 0.02.

2. Photoelectric Matrix Elements

To the photoelectric matrix elements in Table I, the form factor F makes no substantial contribution. It ranges from 0.96 for the $\Delta(1236)$ to 0.78 for the $F_{15}(1688)$. As it is, we see that our relativistic result is close to Walker's nonrelativistic result[6] – so that one could say only that our relativistic analysis confirms the choices made in the nonrelativistic model and has eliminated any effective free parameters (other than the observed masses and Ω, of course). The agreements shown here, in particular the small (or zero) values predicted here just when the experiment gives zero, impressed Walker. Some of the results depend on our choice of a harmonic potential, others on the general way the spins and orbits line up. In addition, the special values of F and D predicted by the model are needed for some of these numerical agreements. The model of three spin-$\frac{1}{2}$ quarks in a symmetrical over-all state does seem to be strongly confirmed.

The worst photoelectric matrix element [the $F_{15}(1688)$, helicity $+\frac{3}{2}$ p] in Table I is off by a factor of 2.3. We see no excuse for this as only one term contributes (the magnetic-moment interaction does not contribute here). Most of our meson-width calculations give amplitudes within a factor 1.4 where there is no cancellation of interfering terms. The $S_{11}(1535)$, helicity $+\frac{1}{2}$ p, is off by 1.6 and is outside these limits also.

The sign of a single matrix element depends on the sign of the wave function and may be chosen arbitrarily. The signs given by Walker as coming from experiment really tell the sign of the resonant scattering amplitude relative to a background Born approximation (as explained in more detail in Ref. 6). This total resonant amplitude depends theoretically on the product of two factors: One is the amplitude for absorption of a photon in going to a resonance, and the other the amplitude for the emission of a π by the resonance. Thus the sign of the interference depends on the product of two signs: (1) the sign of the current matrix element $\mathfrak{M}^V$, and (2) the sign of the meson-emission matrix element $\mathfrak{M}^P$. The arbitrary choice of wave-function sign for the intermediate resonance cancels out. Our signs for A are products of three factors: first the over-all sign to agree with the convention of Walker, next the sign calculated for the matrix element of $\mathfrak{M}^V$ from our formulas and conventions (this latter has no absolute significance), and third, the sign of the meson amplitude computed later. The product of all these, as given in Table I, is to be compared to Walker's experimental values.

Included in Table I is one meson photoelectric matrix element, that for $\omega \to \pi + \gamma$. We obtain it from experiment from its branching ratio in ω decay. We cannot determine its sign. The good agreement here means that we are dealing adequately with the mass factors. For example, inclusion of the factor g^2 coming from spins is 1.9 in this case, so replacing F by g^2 would raise the theoretical results by a factor 2.2. This is the same type of evidence (that g^2 is not there) that we noticed in the normalization of the $K\pi e\nu$ decay, but here the effect is larger, and we are not concerned with uncertainties in the Cabibbo angle.

One radiative decay of a hyperon resonance is known, $\Lambda(1520) \to \Lambda\gamma$. Individual helicity amplitudes have not been measured; hence $A^{\exp}$ is to be compared to the square root of the sum of the squares of the theoretical amplitudes.

3. Rates for Pseudoscalar Meson Emission

In order better to discuss and understand the large number of figures in Tables IV and V, we have made a histogram of the results, Fig. 1, plotting the number of cases against the ln of the ratio of theoretical to experimental rates. The baryons are shaded squares, the mesons are open squares.

There is a peak near the center, from -0.6 to $+0.8$, containing $\frac{3}{4}$ the cases, with the other $\frac{1}{4}$ widely spread about, two cases at $+4.4$ and at -5.5 being completely off the scale of the drawing. The peak is not centered but the center lies near $+0.1$. We can move our axis there by renormalizing all our theoretical matrix elements down by choosing a new f_R, say,

$$f_R' = 0.95 f_R,$$

meaning that the predicted nucleon-coupling matrix is off to some extent, just like all the others. Doing this, we see that our bunch around the new center represents a spread of -0.7 to $+0.7$ in the logarithm, i.e., that $\frac{3}{4}$ of the calculated partial widths are within a factor of 2 of the measured partial widths.

In order to get some idea as to whether agreement as poor as ± 0.7 in the ln was significant, we made several observations. First, the actual widths vary from a few tenths to a few hundred MeV, or over a range of 6 in the logarithm. Then we tried changing identifications of particles [e.g., by supposing a $^2(\underline{8})$ was a $^4(\underline{8})$, or a Σ was in a decimet instead of an octet], thus changing the F/D ratios. Such changes made large changes in the ln; ± 2 was not uncommon and much larger (as well as many smaller) values occurred. We concluded that the range ± 0.7 was really very narrow and the agreements with the model were of significance.

The coupling constant f_R' that we eventually choose is not the normal coupling constant, say f_T, expected from the PCAC theory [that theory gives $f_T = (0.08)^{1/2}/m_\pi g_A$, where g_A is the axial-vector coupling constant of the nucleon, 1.22] but differs by a factor, say Z, from it, $f_R' = Z f_T$. We could interpret that factor Z as a general renormalization of the axial-vector coupling constant of a quark, and choose it, rather than f_R', as the adjustable constant of the theory, evaluating it from our fit to f_R', as $Z = 0.70$. With this interpretation, the only change made is that the prediction (52) for g_A would now become $g_A = \frac{5}{3} Z = 1.17$.

The next question to which we address ourselves is whether we can learn anything about the charac-

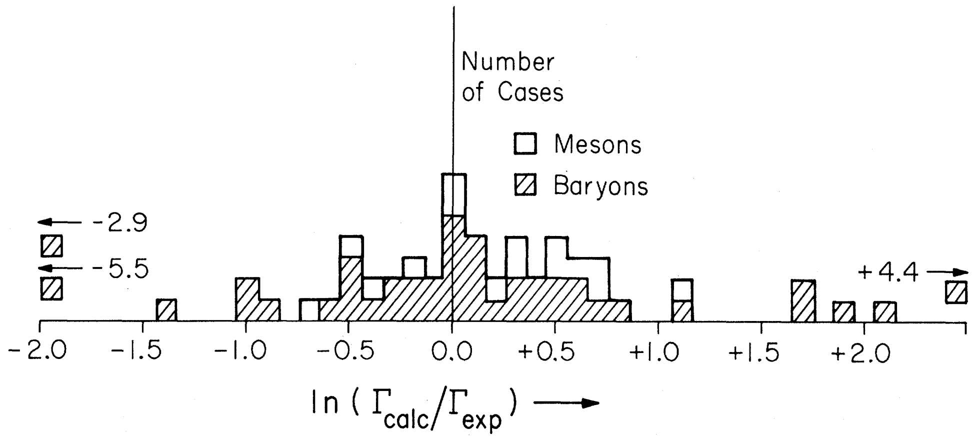

FIG. 1. Histogram of partial widths for pseudoscalar-meson emission.

ter of the model's failings from the nature of the deviations. We list in Table VI all the cases for which the ln is outside the limits -0.6 to $+0.8$. In making this comprehensive table, we have used all the data that are accepted to appear in the Particle Properties Tables[3] (but not in the data card listings, directly). Some of these data are of poor quality and we cannot learn much from a deviation from theory if it is, possibly, experimental. Therefore, in Table VI we have placed an approximate ± on the figure for the ln which is a rough range of experimental widths. Cases in which this range could bring us into the region -0.6 to $+0.8$ do not represent anything of significance. These cases have been listed first and are all those before we get to

TABLE VI. List of calculated transitions which seriously disagree with experiment.

State	J^P	Mode	$\ln(\Gamma_{calc}/\Gamma_{exp})$	β term
$K^*(1420)$	2^+	ωK	-0.7 ± 0.6	
$\Sigma(1765)$	$\frac{5}{2}^-$	$\Sigma^*\pi$	-0.9 ± 0.5	
$\Delta(1890)$	$\frac{5}{2}^+$	$N\pi$	-1.0 ± 0.5	
$N(1700)$	$\frac{1}{2}^-$	$N\pi$	-1.4 ± 0.7	β
$\Sigma(2030)$	$\frac{7}{2}^+$	$\Sigma\pi$	$+1.7\pm0.7$	
$N(1535)$	$\frac{1}{2}^-$	$N\pi$	$+1.7\pm0.6$	β
$\phi(1019)$	1^-	$K\bar{K}$	$+1.1\pm0.1$	
$\Lambda(1690)$	$\frac{3}{2}^-$	$N\bar{K}$	$+1.9\pm0.6$	
$\Sigma(1915)$	$\frac{5}{2}^+$	$N\bar{K}$	-1.0	
		$\Lambda\pi$	$+1.1$	
		$\Sigma\pi$	$+2.1$	
$N(1470)$	$\frac{1}{2}^+$	$N\pi$	-2.9 ± 0.4	β
$\Lambda(1670)$	$\frac{1}{2}^-$	$N\bar{K}$	$+4.4\pm0.7$	β
$N(1780)$	$\frac{1}{2}^+$	$N\pi$	-5.5 ± 0.6	β

$\Sigma(2030)\to\Sigma\pi$, which is marginal, as is $N(1535)\to N\pi$.

Also listed in Table VI is a reminder as to whether the formula for the matrix elements involves a term in β, or whether only $\gamma\lambda$ and $\gamma\lambda\delta$ (δ is positive near 0.1, roughly) come in. These β terms subtract from the other terms and an interference between two terms is involved.

For the $\Lambda(1690)\to N\bar{K}$ we have no explanation and are unable to learn anything from it. No cancellation of β terms is involved and many similar disintegrations work quite adequately in other multiplets.

We have three apparently serious very large deviations. They are all states of spin $\frac{1}{2}$ and involve cancellation of two terms. Such a situation is a sensitive test of the accuracy of the calculations. Take the $\Lambda(1670)\to N\bar{K}$ at $+4.4$, for example. The theoretical value is much too big. May that mean that the experimental value is, in a sense, too small? Theoretically, two terms are being subtracted; they are three to one in ratio. If they were in reality closer together, so that in nature they nearly cancelled, then our theoretical result would be much too high. Again, in reverse, for $N(1780)\to N\pi$ at -5.5, the two terms theoretically cancel almost exactly. If in nature the balance were not so perfect, the rate could be much higher. It is also possible that we have misidentified this state. The Roper resonance $N(1470)$ has a logarithmic error of 2.9, which may be due to the effect of cancellations also. However, we must remember that this orbital state requires a special mass-squared correction of -0.82 GeV^2 before it can be compared to the rest of the particles. Such a serious effect might well change the wave functions, and matrix elements, in a serious way. We might well question whether the oscillator model is correctly describing these spin $\frac{1}{2}$

states. In particular, the wave function of excited states of zero total orbital angular momentum cannot be trusted.

The other possibility is that the matrix element $\mathfrak{M}^P$ for pseudoscalar mesons is wrong. If, for example, we start with just γ_5 (or $\gamma_5 \not{q}$) taken between the spinors (instead of $\not{p}_2\gamma_5\not{q}+\gamma_5\not{q}\not{p}_1$), the coupling arises without a β term entirely. This is very unsatisfactory, because many other states that previously were all right go seriously awry [the $N(1535)\to N\eta$ can hardly occur at all, the $A_1(1070)$ and $B(1235)$ do not show a reasonable angular distribution, etc.], but it does show that the precise form of the β term is sensitive to our assumptions.

It is not easy to find a prescription to describe these deviations systematically.

The β term arises in nonrelativistic language as follows. In first approximation, a pseudoscalar meson couples with $\vec{\sigma}\cdot\vec{Q}$ and thus, as Q falls, so does the matrix element. But take the $N(1535)$, for example. In fact, we shall have two powers of Q, one from the $\vec{\sigma}\cdot\vec{Q}$ and one from the retardation factor $e^{iq\cdot u_a}$, since the π must come out in an S wave. So, for small Q, there is only a very small amplitude for disintegration. If there were some way its spin could couple to the velocity of motion of the interior orbits instead of to the pion momentum $\vec{Q}$, it could use this momentum to form the pseudoscalar with $\vec{\sigma}$ and have a finite amplitude to emit as an S wave. Spin-orbit coupling would distort the wave functions so as to make this possible, but we tried to keep the theory simple and with few arbitrary constants. When coupling nonrelativistically to a particle of $\vec{P}$ and M, $\vec{\sigma}\cdot\vec{Q}$ is replaced by $\vec{\sigma}\cdot(\vec{Q}-\vec{P}\nu/M)$, where ν is the frequency of the meson, for $\vec{Q}$ is not even a Galilean invariant while $\vec{Q}-\vec{P}\nu/M$ is. If $\vec{P}$ is the external momentum of the states, this has no effect in the c.m. system. But if each quark couples in this way, with $\vec{P}$ being its individual internal momentum, then we have a direct way to emit S waves for small Q. This has been pointed out by Mitra.[1] Our expression (42) does correspond nonrelativistically to $\sum e'_a\vec{\sigma}_a\cdot(\vec{Q}-\vec{P}_a\,\nu/M)$, where M is "a quark mass" $\frac{1}{3}$ the mass of one of the states.

The $\Sigma(1915)\ {}^2(8)_{5/2}$ has all three of its modes in the list. The deviations are in opposite directions and experimental uncertainties do not seem to be large enough to account for anything like this. [*Note added in proof:* A. Barbaro-Galtieri has pointed out to us that this pattern could result from a single experimental number being high, the rate $\Gamma_{N\bar{K}}$ for $\Sigma(1915)\to N\bar{K}$, since the other two partial widths are determined from data giving their product with $\Gamma_{N\bar{K}}$.] It is this particle that does not fit well in our mass scheme. We suggest that it has been misidentified by us and belongs in the decimet ${}^4(\underline{10})_{5/2}$ instead of in the octet (even though its mass is even more unsatisfactory there). The rates for all three decay modes $N\pi$, $\Lambda\pi$, $\Sigma\pi$ fit somewhat better [$\ln(\Gamma_{\rm th}/\Gamma_{\rm exp})$ becomes -0.9, -0.2, -0.7, respectively].

In calculating some of the rates of Table V, where two pseudoscalar mesons are emitted, we are faced with a serious shortcoming of our theory. For instance, in the decay $K^*(892)\to K\pi$, the result is not symmetric with respect to whether the π or the K is replaced via PCAC by the divergence of the axial-vector current, the other meson being a state of two quarks. We therefore list both cases, $K^*\to K\pi$ and $K^*\to\pi K$, the last meson being the one replaced by the axial-vector current. For the heavier mesons η and η', the situation is even worse. Using the K current always gives the higher value (by 0.7 in ln). When this theory is ultimately made more symmetrical, the πK rates will be reduced, and possibly the $\phi\to K\bar{K}$ rate will be reduced somewhat also. This makes it look like the somewhat $\phi\to K\bar{K}$ rate is another sign of this need for another formulation. There is no numerical evidence that this difficulty extends to the baryons. The mean of $\ln(\Gamma_{\rm calc}/\Gamma_{\rm exp})$ for K-emitting decays (leaving out the two at +1.9 and +4.4) is also at +0.1, just like the mean of the π-emitting decays.

We probably also have a different symmetry problem in a decay involving a vector meson (like $A_2\to\pi\rho$), for we could also expect to be able to replace the vector meson by a vector current. We have not calculated these cases that way.

For the $A_1(1070)$ and $B(1235)$, some measurements of the ratio of the helicity +1 to helicity 0 amplitudes have been made. We get for these ratios

$$A_1(1)/A_1(0)=0.76,\quad B(0)/B(1)=0.19.$$

Two experiments[12] for the A_1 give 0.5 or 0.9, respectively, while for the $B(1235)$ measured values[12] vary from 0.2 to 0.7.

DISCUSSION OF RESULTS

We emphasize again that we do not think we have a theory in the ordinary sense, but something more akin to curve fitting, in which a simple mathematical function is fitted to an empirical curve without an implication that this function is an ultimately correct or even a rational "explanation" of the shape of the curve. It is only a description, useful to keep in mind as a way of remembering data when attempting to find a more satisfactory understanding of that data. So, in this spirit, the equations we have introduced and the quark quantum numbers belonging to them have been used to fit data.

The most striking aspect of the agreement does imply, we think, that the quantum numbers of the symmetrical quark model with internal motion (or its analog) lead to a good representation of the kinds of multiplets that are found. Most particularly, it yields, to a good approximation, the F/D ratios expected of these multiplets in various interactions. Although it fails for several states of $J=\frac{1}{2}$, these states yield matrix elements which are the difference of two terms, each of which varies with mass and momentum as we go from one state of the multiplet to another, thus explaining the utterly erratic results if one tries to fit the over-all matrix element by any F/D ratio at all.[13]

The fact that the over-all rate constants vary correctly as we pass from multiplet to multiplet with the same N seems to show that the "internal"-motion quantum momentum variables and the spin do combine like two (for mesons, one) orbital angular momenta and a spin angular momentum.

We cannot conclude much from the fit as we vary the degree of excitation N; nor, therefore, have we tested the harmonic-oscillator character of the wave functions with any precision. This is because most of our results for baryons correspond to only three values of N, $N=0,1,2$ (with only one state at $N=4$), and for the mesons only to two, $N=0,1$. Further, we have an arbitrary adjustment factor which varies a great deal as we vary N, so little has been tested here. We cannot consider the equal squared-mass spacing rule, for it is this rule which prompted the harmonic-oscillator assumption in the first place.

The reader should also remember that there are, in addition, two well-known points which show that a quark model of resonances is on the right track. First, baryon resonances that cannot be formed of three quarks, or meson resonances that cannot be made of a single quark-antiquark pair, should not be found. This implies that "exotic" resonances for SU_3 multiplets outside the $\underline{1}$, $\underline{8}$, and $\underline{10}$ should not be found. There are some suggested by data, but the question is still being debated. Again, for mesons it is impossible with $q\bar{q}$ to form certain combinations of J^P and charge conjugation C (such as 0^- with $C=-1$ or $J^P=0^+,1^-,2^+,\ldots$ with $CP=-1$), and none of these are found.

The second point which gives evidence that the currents are those carried by symmetrical quarks is the $\Delta I=\frac{1}{2}$ rule for weak, nonleptonic hyperon decays.[14] Empirically, the nonleptonic strangeness-changing decays of the hadrons obey a simple rule, that they are nearly purely of isospin change $\frac{1}{2}$; whereas the theory of weak interactions, as usually interpreted, expects $\Delta I=\frac{3}{2}$ in roughly equal measure.

The weak-interaction theory[15] represents the interaction as the direct interaction of two currents, $J_\mu{}^\dagger J_\mu$, where the current J_μ is a sum of terms, the hadronic part being the sum of two terms, one changing strangeness (of strength $\sin\theta$) and one not (of strength $\cos\theta$, where θ is the Cabibbo angle). The cross term between these two is the origin of this interaction. In the quark model, this would be represented as

$$H_{\text{int}}=\sin\theta\cos\theta\,(\bar{\psi}_1\gamma_\mu a\psi_2)(\bar{\psi}_3\gamma_\mu a\psi_4)\,, \tag{56}$$

where 1 is a strange quark s, and 2, 3, and 4 are nonstrange quarks of type u, u, and d, respectively, and a is $1+i\gamma_5$. It is natural to suppose that (56) acts among the three quarks in a totally symmetric wave function, in a baryon.

But, in the approximation that all the interactions take place at the same point, the form (56) is antisymmetrical for quarks 2 and 4, so that

$$H_{\text{int}}=-\sin\theta\cos\theta\,(\bar{\psi}_1\gamma_\mu a\psi_4)(\bar{\psi}_3\gamma_\mu a\psi_2)$$

This means that the two nonstrange quarks 2 and 4 will only interact if they are in a state antisymmetrical in their Dirac spinors. They are symmetrical in space, being at the same point, so *by virtue of the assumption that they are totally symmetric within the baryon,* they must be antisymmetric in unitary spin if they are to enter into weak interaction. In this case, the unitary spin is simply the isospin and the antisymmetric isospin state has $I=0$. The strange quark has zero isospin, so the outgoing quarks (1, 3) have isospin $\frac{1}{2}$. Hence, only $\Delta I=\frac{1}{2}$ occurs. For non-strangeness-changing decays, we find $\Delta I=0,1$. The coupling would be a member of an SU_3 octet.

In an interaction between a quark and antiquark (such as might arise in the simplest model of a meson), expression (56) can contribute to diagrams of the two types illustrated in Fig. 2. If the relative sign of these two diagrams is positive, as appropriate to Bose particles, the rule $\Delta I=\frac{1}{2}$ results again.

The $\Delta I=\frac{1}{2}$ rule is not perfect; $\Delta I=\frac{3}{2}$ does occur,

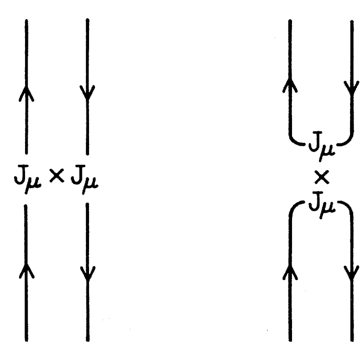

FIG. 2. Weak currents acting on q and $\bar{q}$.

roughly 500 times less frequently. This may result from the fact that the interaction is not quite pointlike; perhaps it results from an intermediate W particle of large, but not infinite, mass. It also may be a consequence of quarks which are not completely bosons, but partially antisymmetric, for example in different baryons.

To summarize, the $\Delta I=\frac{1}{2}$ rule may be evidence that the current-carrying spin $\frac{1}{2}$ elements of the weak hadronic interaction obey symmetrical statistics.

Lest the reader by carried away by the apparently good fits, we remind him of the severe limitations of the model as a possible ultimate "theory." There is first of all the lack of unitarity associated with the excitations of the oscillators in timelike directions. There is an analogous set of extra states arising from the spins. The couplings to pseudoscalar mesons, via the divergence of axial-vector current, leads to results that are unsymmetrical for such decays as $K^* \to K+\pi$ (the result depends upon which final mesons is replaced by the divergence of the current). The expected electromagnetic form factor of the proton, and the deep-inelastic electron-proton scattering as well, are hopelessly at variance with observations. There still are at least three $J=\frac{1}{2}$ baryon partial decay widths that are in severe disagreement with experiment. Finally we must remember that the $\Lambda(1405)$ and the meson $L=1$ multiplets do not confirm the assumed lack of spin-orbit coupling, and the $\Sigma(1915)$ does not conform to the expected mass regularities.

REMAINING PROBLEMS

If we continue to discuss this "model" from the point of view of curve fittings, the wider the range of problems to which the model may be applied with as little arbitrariness as possible, the more useful it is. It would be desirable to have a model such that almost anything can be calculated; even if it turns out to be wrong it can at least be compared to any experiment. Our model is not yet formulated so that this can be done. We can deal with interactions with currents in first order, as we have done, or in higher order [as $(\delta K)K^{-1}(\delta K)$]; but there are a number of problems whose formulation is not at all obvious.

Firstly, we have no relation of the dynamics of meson and baryon and so we cannot, for example, show that the Regge slopes are the same for each of them (we assumed it). Again we do not see how to proceed, by perturbation series or what, to calculate more accurate masses for the particles. That is, the contributions to C in Eq. (1), and analogous constants for the mesons, are left outside the ability of the model to calculate. In the same way, no precise obvious plan is available to calculate such things as the spin-orbit coupling or the Σ-Λ mass differences.

In the nonrelativistic quark model, which is a model close to ours, the annihilation of the quark and antiquark in a meson into a current (for such processes as $\pi\to\mu+\nu$ or $\rho\to$virtual photon$\to e^+e^-$) seems straightforward and would not involve any new arbitrary constant. In our present model, we would guess that such an amplitude is just proportional to the matrix element of the current (18) sandwiched between the quark and the antiquark, or

$$\langle i\,|j_\mu^V|\mathrm{vac}\rangle = a e_a \int \mathrm{Tr}[\,h_i(p_a,p_b)(\not{p}_a\gamma_\mu e^{iq\cdot x_a}-\gamma_\mu e^{iq\cdot x_b}\not{p}_b)]\times d^4(p_a-p_b)/(2\pi)^4\,, \tag{57}$$

where h_i is the eigenfunction of the operator $\mathfrak{N}$ for the meson in question of four-momentum $q=p_a+p_b$ and mass m $(q^2=m^2)$, and a is a constant. We find

$$\langle\rho^0\,|j_\mu^V|\mathrm{vac}\rangle = e_\mu\frac{\Omega}{\pi}m_\rho a = e_\mu g_\rho m_\rho\,, \tag{58}$$

where e_μ is the polarization of the ρ, and for the π^0,

$$\langle\pi^0\,|j_\mu^A|\mathrm{vac}\rangle = q_\mu\frac{\Omega}{\pi}a = q_\mu f_\pi\,. \tag{59}$$

The result for the K is the same with a different component of the current, as predicted by SU_3, defining a corresponding f_K. Experimentally,

$$f_\pi = 95\ \mathrm{MeV}\,,\quad f_K=109\ \mathrm{MeV}\,,$$
$$g_\rho=160\ \mathrm{MeV}\,,\quad 3g_\omega=156\ \mathrm{MeV}\,,\quad (3/\sqrt{2})g_\phi=168\ \mathrm{MeV}\,,$$

where we would expect them to be equal. If we adopt the view that the axial-vector coupling of the quark is renormalized, the first two numbers are replaced by $f_\pi/Z=136$ MeV, and $f_K/Z=156$ MeV with $Z=0.70$.

Why can we not determine the constant a absolutely from theory, and by what theoretical view do we justify the form (57) in our model?[15a] This question is important also because if a could be theoretically deduced, then using the Goldberger-Treiman relation we could have a prediction of the absolute size of f_R, the coupling constant of mesons to hadrons. It is tantalizing that such a process has a definite absolute rate in the nonrelativistic model (but it gives an a which is not constant but varies inversely with the mass of the state, a feature completely at variance with experiment).

Again, in the nonrelativistic model, such things as the electromagnetic self-energy differences $p-n-\Sigma^++\Sigma^0$, $\Sigma^++\Sigma^--2\Sigma^0$, or $\pi^+-\pi^0$ could be calculated since they depend only on the electrical interaction between quarks. The latter can be computed from the wave function, and thus absolute values can be computed. In our model it is not obvious how to do this. Our prescription would be via the second-order scattering theory; take one vector interaction of momentum q, the other of momentum $-q$, multiply by $4\pi e^2/q^2$, and integrate over $d^4q/(2\pi)^4$. It is rather complicated, and one does not have the feeling it is surely right. We have not carried it out.

There is another problem showing inadequacies in the formulation of the theory of our model. At high energy, total cross sections approach their asymptotic values as const $+\beta s^\alpha$, with $\alpha(t=0)$ about -0.5 or so for the ρ and A_2 trajectories. This agrees with mass formulas, of course, but the coefficients β for the various kinds of mesons and nucleons obey some very simple relationships closely related to those expected in a picture using quarks. Can we calculate the $\beta(t)$ expected in our model?

We can do so by summing the terms corresponding to exchange of the sequence of mesons belonging to the trajectory (increasing the values of L and J together). (For the case of baryon resonances, the sequence is that formed by adding successively two units of angular momentum in the combination $a^*a^*+b^*b^*$.) But we cannot calculate the exchange of any meson except for the lowest S states. This is because we have not specified how to calculate a hadron-hadron-meson vertex for an arbitrary meson – only for the pseudoscalar and vector mesons have we stated what to do (replace by currents).

An obvious choice for this interaction is illustrated in Fig. 3 for the case of three mesons. Here, lines are drawn indicating the direction of the quark and the antiquark in each meson. The coupling, disregarding spin and unitary spin for a moment, would be written

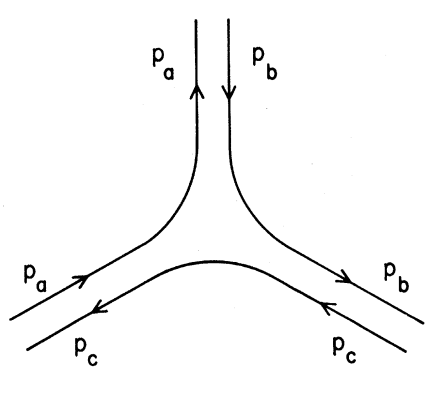

FIG. 3. Quark momenta in the three-meson vertex.

$$\int h_1(u_c,u_b)h_3(u_b,u_a)h_2(u_a,u_c)d^4u_a d^4u_b d^4u_c\,, \qquad (60)$$

where h_i are the wave functions for each meson. Of course, the center-of-mass motion can be extracted out, obtaining the conservation of over-all momentum ($p_1+p_2+p_3=0$), and leaving only two internal-variable integrations. To include unitary spin, each h_i is thought of as a 3×3 matrix in unitary spin, with the entering index for the quark, and the second index for the antiquark. The same would hold for the Dirac spinor indices; the antiquark would carry an adjoint spinor index. Thus the h_i's are matrices; for the diagram shown they are to be multiplied and the trace taken.

This does make the couplings very much like the current couplings we have used, but not quite in one essential way. If it is assumed, as we have, that the spinors of h_i are in the direction of its proper motion P, then what obtains for pseudoscalar mesons is an expression close to Eq. (42) but with the term β missing. Such a coupling fits the $\frac{1}{2}$ states still more poorly. One possibility is to complicate (60) by inserting in each junction the operator $\not{p}_a$, where $p_{a\mu}$ is the quark momentum. Thus (60) would become

$$\int \mathrm{Tr}[\not{\nabla}_c h_1(u_c,u_b)\not{\nabla}_b h_3(u_b,u_a)\not{\nabla}_a h_2(u_a,u_c)]$$
$$\times d^4u_a d^4u_b d^4u_c\,. \qquad (61)$$

Such an expression is complicated and *ad hoc* and we have not yet made any investigation of it. We are looking for theoretical arguments to limit our choices. The extension to baryon-baryon-meson presents no new problem.

The development of such expressions as (60) or (61) to give a form for the arbitrary meson coupling that would agree with experiment, would not only permit extending our theory to Regge couplings $\beta(t)$, but would solve a number of our present problems as well, of course. For example, it would be symmetrical in the mesons. In addition, it would permit us to analyze a current coupling via an intermediate meson and thus to have hopes of producing a more realistic electromagnetic form factor for the proton. Again, the Goldberger-Treiman relation may emerge as a consequence of a dynamical model, instead of being put in from the beginning. Finally, the interaction of the quark and antiquark in a meson branch might in some natural way be related dynamically to the interaction they have within the baryon, so some understanding of the equality of Regge slopes for mesons and baryons might be contained in the model.

A word about the relevance of our work to work following the Veneziano line[16] is in order. There the meson, for example, has not one internal oscillator of frequency Ω, but a whole sequence of modes of frequency Ω, 2Ω, etc. The actual low states that we have studied will be almost the same, however, since the first levels above ground, at energy Ω, can come only from the excitation of the one oscillator mode of frequency Ω. This includes all the meson states we studied, and very many of the baryon states. The next level at 2Ω, in baryons, can be made of two oscillations of the first mode, as we assumed. But there is a new additional $\underline{70}$ of negative parity from the first excitation of the second mode (at 2Ω). Thus all the states of even parity near 2Ω have been correctly assigned – only a very few states of high energy may get a new assignment, so that virtually all of our work is relevant to the question of the numerical agreement of theories of this type to experiment.

The quark model in another form has proved useful in understanding certain aspects of high-energy collisions[17] (other than Regge behavior). It is not yet clear how our low-energy model should be used at high energy in a consistent way, but it is an indication that a quark picture may ultimately pervade the entire field of hadron physics.

About the paradoxes of the quark model we have nothing to add, except perhaps to make these paradoxes more poignant by exhibiting the mysteriously good fit of such a peculiar model. Can we avoid complications like parastatistics, with its implication that there are three kinds of each quark, and that Δ^{++} is composed of three u quarks, one of each kind? But, if quarks are identical bosons, how could baryons be fermions? If they are only bosons inside the baryon, what happens if they come apart – must they not be purely fermions as the spin-and-statistics theorem suggests? Or is that theorem based on the assumption that quarks can appear singly? Perhaps they cannot be separated because the harmonic-oscillator potential rises as they get farther apart – or better, perhaps the harmonic-oscillator potential is an artifact (like the magnetic exchange force in iron) reflecting in a crude way the proposition that individual quarks cannot be separated because of their statistics.

APPENDIX: THE WAVE FUNCTIONS OF THE BARYON STATES

We shall discuss the calculations for baryons; those for mesons are similar but simpler.

1. The Wave-Function Symmetries of Three Objects

If an object can be in one of a number of conditions x, y, z, ... we can, when we have three such objects, form states of four kinds of symmetry which we call S, α, β, A, symmetric (S), mixed-symmetric (α, β), and antisymmetric (A),

$$\begin{aligned}
|S\rangle &= |xyz\rangle_S = \frac{1}{\sqrt{6}}(|xyz\rangle + |xzy\rangle + |yxz\rangle \\
&\qquad + |yzx\rangle + |zxy\rangle + |zyx\rangle)\,, \\
|\alpha\rangle &= |xyz\rangle_\alpha = \frac{1}{2\sqrt{3}}(|xyz\rangle + |xzy\rangle + |yxz\rangle \\
&\qquad + |yzx\rangle - 2|zxy\rangle - 2|zyx\rangle)\,, \\
|\beta\rangle &= |xyz\rangle_\beta = \tfrac{1}{2}(|xyz\rangle - |xzy\rangle + |yxz\rangle - |yzx\rangle)\,, \\
|A\rangle &= |xyz\rangle_A = \frac{1}{\sqrt{6}}(-|xyz\rangle + |xzy\rangle - |yzx\rangle \\
&\qquad + |yxz\rangle - |zxy\rangle + |zyx\rangle)\,,
\end{aligned} \tag{A1}$$

where $|zxy\rangle$ means that the first object is in state z, the second in x, and the third in y. If, say, x, and y are the same state $y=x$, we must replace $|xyz\rangle + |yxz\rangle$ by $\sqrt{2}|xxz\rangle$. If x, y, z are all the same, only the S state survives as $|xxx\rangle_S = |xxx\rangle$. The state α has been chosen to be symmetric in the last two quarks, the state β is antisymmetric. If we combine two states of these kinds, say $|1\rangle$, and $|2\rangle$, we may recompose states of varying symmetry by the following rules:

$$\begin{aligned}
&|1\rangle_S|2\rangle_S = |\,\rangle_S\,, \quad |1\rangle_S|2\rangle_\alpha = |\,\rangle_\alpha\,, \\
&|1\rangle_S|2\rangle_\beta = |\,\rangle_\beta\,, \quad |1\rangle_S|2\rangle_A = |\,\rangle_A\,, \\
&|1\rangle_A|2\rangle_S = |\,\rangle_A\,, \quad |1\rangle_A|2\rangle_\alpha = |\,\rangle_\beta\,, \\
&-|1\rangle_A|2\rangle_\beta = |\,\rangle_\alpha\,, \quad |1\rangle_A|2\rangle_A = |\,\rangle_S\,, \\
&\frac{1}{\sqrt{2}}(+|1\rangle_\alpha|2\rangle_\alpha + |1\rangle_\beta|2\rangle_\beta) = |\,\rangle_S\,, \\
&\frac{1}{\sqrt{2}}(-|1\rangle_\alpha|2\rangle_\alpha + |1\rangle_\beta|2\rangle_\beta) = |\,\rangle_\alpha\,, \\
&\frac{1}{\sqrt{2}}(+|1\rangle_\alpha|2\rangle_\beta + |1\rangle_\beta|2\rangle_\alpha) = |\,\rangle_\beta\,, \\
&\frac{1}{\sqrt{2}}(-|1\rangle_\alpha|2\rangle_\beta + |1\rangle_\beta|2\rangle_\alpha) = |\,\rangle_A\,.
\end{aligned} \tag{A2}$$

2. Dependence on Spin

If we combine spins so that x, y, and z must either be $+\frac{1}{2}$ or $-\frac{1}{2}$ (written simply +, −), we find that $|\,\rangle_S$ is spin $\frac{3}{2}$. $|\,\rangle_{\alpha,\beta}$ are spin $\frac{1}{2}$ and $|\,\rangle_A = 0$. We have four states of spin $\frac{3}{2}$,

$$\begin{aligned}
&|\tfrac{3}{2}, +\tfrac{3}{2}\rangle_S = |+++\rangle_S\,, \\
&|\tfrac{3}{2}, +\tfrac{1}{2}\rangle_S = |++-\rangle_S\,, \\
&|\tfrac{3}{2}, -\tfrac{1}{2}\rangle_S = |+--\rangle_S\,, \\
&|\tfrac{3}{2}, -\tfrac{3}{2}\rangle_S = |---\rangle_S\,,
\end{aligned} \tag{A3'}$$

and two α states of spin $\frac{1}{2}$,

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle_\alpha = +|++-\rangle_\alpha ,$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle_\alpha = -|--+\rangle_\alpha , \tag{A3$''$}$$

and the corresponding β states,

$$|\tfrac{1}{2}, +\tfrac{1}{2}\rangle_\beta = +|++-\rangle_\beta ,$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle_\beta = -|--+\rangle_\beta . \tag{A3$'''$}$$

3. Dependence on Unitary Spin

If the objects can have three values for unitary spin, we have several possibilities. $|\,\rangle_S$ is a decimet, $|\underline{10}\rangle_S$; α and β are octets, $|\underline{8}\rangle_\alpha$, and $|\underline{8}\rangle_\beta$; and $|\,\rangle_A$ is a singlet, $|\underline{1}\rangle_A$. The wave functions for these are evident. For example, the quantum numbers for a Σ^0 require it to be made of an s, u, d quark. Therefore Σ^0 in a state $|8\rangle_\alpha$ is $|s, u, d\rangle_\alpha$, in a decimet it is $|s, u, d\rangle_S$, where we mean to substitute into (A1) s, u, d for x, y, z, respectively. A neutron is d, d, u, so if it is in an octet it is $|d, d, u\rangle_{\alpha,\beta}$, if it is in a decimet it is $|d, d, u\rangle_S$ and is called a Δ^0, etc.

Combining these two (spin and unitary spin) by our multiplication table (A2), we can make the following $\underline{56}$ symmetrical states $|56\rangle_S$ [we use "quartet" and "doublet" symbols $^4(\ )$ and $^2(\ )$ to represent total spin $\frac{3}{2}$ and $\frac{1}{2}$ states; there is no $|\text{spin}\rangle_A$]:

$$|\underline{56}\rangle_S\!: \ {}^4(\underline{10}) = |\tfrac{3}{2}\rangle_S\,|\underline{10}\rangle_S ,$$
$${}^2(\underline{8}) = \frac{1}{\sqrt{2}}(\,|\tfrac{1}{2}\rangle_\alpha|\underline{8}\rangle_\alpha + |\tfrac{1}{2}\rangle_\beta\,|\underline{8}\rangle_\beta) ; \tag{A4}$$

the following $\underline{70}$ states of type α:

$$|\underline{70}\rangle_\alpha\!: \ {}^4(\underline{8})_\alpha = |\tfrac{3}{2}\rangle_S\,|\underline{8}\rangle_\alpha ,$$
$${}^2(\underline{10})_\alpha = |\tfrac{1}{2}\rangle_\alpha\,|\underline{10}\rangle_S ,$$
$${}^2(\underline{8})_\alpha = \frac{1}{\sqrt{2}}(-|\tfrac{1}{2}\rangle_\alpha\,|\underline{8}\rangle_\alpha + |\tfrac{1}{2}\rangle_\beta\,|\underline{8}\rangle_\beta) , \tag{A5}$$
$${}^2(\underline{1})_\alpha = -|\tfrac{1}{2}\rangle_\beta\,|1\rangle_A ,$$

and $\underline{70}$ of type β:

$$|\underline{70}\rangle_\beta\!: \ {}^4(\underline{8})_\beta = |\tfrac{3}{2}\rangle_S\,|\underline{8}\rangle_\beta ,$$
$${}^2(\underline{10})_\beta = |\tfrac{1}{2}\rangle_\beta\,|\underline{10}\rangle_S ,$$
$${}^2(\underline{8})_\beta = \frac{1}{\sqrt{2}}(\,|\tfrac{1}{2}\rangle_\alpha\,|\underline{8}\rangle_\beta + |\tfrac{1}{2}\rangle_\beta\,|\underline{8}\rangle_\alpha) , \tag{A6}$$
$${}^2(\underline{1})_\beta = |\tfrac{1}{2}\rangle_\alpha\,|\underline{1}\rangle_A ;$$

and the following $\underline{20}$ antisymmetric states:

$$|\underline{20}\rangle_A\!: \ {}^4(\underline{1})_A = |\tfrac{3}{2}\rangle_S\,|\underline{1}\rangle_A ,$$
$${}^2(\underline{8})_A = \frac{1}{\sqrt{2}}(-|\tfrac{1}{2}\rangle_\alpha\,|\underline{8}\rangle_\beta + |\tfrac{1}{2}\rangle_\beta\,|\underline{8}\rangle_\alpha) . \tag{A7}$$

4. Dependence on Space

Next we must work out the orbital states and simply combine them with these SU_6 states to produce purely over-all symmetrical states, for we assume the baryon resonances are pure $|\,\rangle_S$ states only.

$N=0$. The ground state, h_0, is symmetric. We call it $|g\rangle$. Therefore, it combines only with $|\underline{56}\rangle_S$ to make a totally symmetrical state,

$$|\underline{56}, 0\rangle = |\underline{56}\rangle_S\,|g\rangle , \tag{A8}$$

yielding ${}^4(\underline{10})_{3/2}$ for the spin-$\frac{3}{2}$ decimet and ${}^2(\underline{8})_{1/2}$ for the spin-$\frac{1}{2}$ octet.

$N=1$. The states are $a^*|g\rangle$ or $b^*|g\rangle$. The first is an α state, the second a β state.

For each case, choosing a^* spacelike, we have three states forming the components of an $L=1$ state. If we select those components as having a z component $+1, 0, -1$, we have for these orbital states

$$|1, +1\rangle_\alpha^1 = a_+^*|g\rangle ,$$
$$|1, 0\rangle_\alpha^1 = a_z^*|g\rangle , \tag{A9}$$
$$|1, -1\rangle_\alpha^1 = a_-^*|g\rangle ,$$

where $a_\pm^* = \mp(a_x^* \pm i\,a_y^*)/\sqrt{2}$, and the superscript 1 in the state system is to clarify that it is an orbit state of $N=1$. Corresponding $|1\rangle_\beta$ states are generated by b^*. Symmetric states are now formed by combining

$$|\underline{70}, 1\rangle = \frac{1}{\sqrt{2}}(\,|\underline{70}\rangle_\alpha\,|1\rangle_\alpha^1 + |\underline{70}\rangle_\beta\,|1\rangle_\beta^1) . \tag{A10}$$

Each state, say the ${}^4(\underline{8})$, being obtained by substituting the expression for that state in (A5) for $|\underline{70}\rangle_\alpha$, and in (A6) for $|\underline{70}\rangle_\beta$.

Which components of angular momentum, of spin, and of orbit are to be combined in (A10)? That depends on which total orbital angular momentum you wish to work out. Thus, for the $N(1700)$ ${}^4(\underline{8})_{1/2}$ we must combine the $S=\frac{3}{2}$ in the expressions (A5) or (A6) for ${}^4(\underline{8})$ to the $L=1$ of the orbit states in (A10), to compound a suitable (say, $+\frac{1}{2}$) component of total spin $J=\frac{1}{2}$, using a linear combination with the correct Clebsch-Gordan coefficients for these values of angular momentum. In this way the wave functions for all of the $|\underline{70}, 1\rangle$ states of various J are worked out (they are all of negative parity, because of the a^*).

$N=2$. Here we can have two excitations of orbital motion and thus we can combine them, using the rules (A2) to make up space states of various symmetry. The parity is now positive.

$$N=2:\quad |2\rangle_S^2, |0\rangle_S^2 = \frac{1}{\sqrt{2}}(|1\rangle_\alpha^1|1\rangle_\alpha^1 + |1\rangle_\beta^1|1\rangle_\beta^1)\,,$$

$$|2\rangle_\alpha^2, |0\rangle_\alpha^2 = \frac{1}{\sqrt{2}}(-|1\rangle_\alpha^1|1\rangle_\alpha^1 + |1\rangle_\beta^1|1\rangle_\beta^1)\,,$$

$$|2\rangle_\beta^2, |0\rangle_\beta^2 = \frac{1}{\sqrt{2}}(|1\rangle_\alpha^1|1\rangle_\beta^1 + |1\rangle_\beta^1|1\rangle_\alpha^1)\,, \tag{A11}$$

$$|1\rangle_A^2 = \frac{1}{\sqrt{2}}(-|1\rangle_\alpha^1|1\rangle_\beta^1 + |1\rangle_\beta^1|1\rangle_\alpha^1)\,.$$

Here again, we shall have to use Clebsch-Gordan coefficients to make an $L=2$ or 0 or 1 out of our two $L=1$ pieces. For example, since

$$|2,0\rangle = \frac{1}{\sqrt{6}}(|+1\rangle|-1\rangle + |-1\rangle|+1\rangle + 2|0\rangle|0\rangle)\,,$$

we have

$$|2,0\rangle_\beta^2 = \frac{1}{\sqrt{6}}(|+1\rangle_\alpha^1|-1\rangle_\beta^1 + |-1\rangle_\alpha^1|+1\rangle_\beta^1 + 2|0\rangle_\alpha^1|0\rangle_\beta^1)$$

$$= \frac{1}{\sqrt{6}}(a_+^* b_-^* + a_-^* b_+^* + 2a_z^* b_z^*)|g\rangle\,,$$

$$|2,0\rangle_S^2 = \frac{1}{\sqrt{6}}(|+1\rangle_\alpha^1|-1\rangle_\alpha^1 + \sqrt{2}|0\rangle_\alpha^1|0\rangle_\alpha^1$$

$$+ |+1\rangle_\beta^1|-1\rangle_\beta^1 + \sqrt{2}|0\rangle_\beta^1|0\rangle_\beta^1)$$

$$= \frac{1}{\sqrt{6}}(a_+^* a_-^* + a_z^* a_z^* + b_+^* b_-^* + b_z^* b_z^*)|g\rangle\,.$$

We take $|0\rangle_\alpha^1|0\rangle_\alpha^1$ to mean the double excitation of a_z^* or $a_z^* a_z^*|g\rangle$, but it is normalized, so it is

$$|0\rangle_\alpha^1|0\rangle_\alpha^1 = \frac{1}{\sqrt{2}} a_z^* a_z^*|g\rangle\,,$$

whereas

$$|0\rangle_\alpha^1|0\rangle_\beta^1 = a_z^* b_z^*|g\rangle\,.$$

These new states must now be combined with the unitary states (A4) to (A7) to form, finally, symmetrized states, and the correct Clebsch-Gordan coefficients used to combine substates of various z components of angular momentum together to form states of definite J and J_z.

We need not discuss the higher states, as the principles are always the same. The meson states are self-evident. Having the states, we must now turn to computing the matrix elements of various operators.

5. Evaluation of One-Particle Matrix Elements

The operators, such as j_μ^V in (18) or $i\, q_\mu j_\mu^A$ in (20), that we shall need here are sums of terms each operating on only one quark. Since the wave functions are properly totally symmetrical, we have to take only the first term, acting on the first quark, a, and multiply by 3. Then each matrix $\mathfrak{M}^V$ [Eq. (35), or Eq. (38)] and $\mathfrak{M}^P$[Eq. (42)] is a sum of simple terms, each term of the sum being the product of an operator on the unitary spin, e_a', of one on the ordinary spin (1 or $\vec{\sigma}$), and of one on the orbital excitation of the variable $\xi(a, a^*)$. Because the operator does not act on the second quark oscillator, the operator b^* is not involved in these matrix elements and it always remains in its ground state. Furthermore, α and β states do not mix.

Since the matrix is a product of operators on unitary spin, on ordinary spin, and on orbital motion, and the wave functions have been explicitly written as sums of products of factors for unitary spin, spin, and orbital motion, the procedure for finding matrix elements is straightforward.

We start with the unitary-spin dependence, where the operator is simply e_a, the charge on the first quark (for photoelectric elements). If e_u, e_d, and e_s are the charges carried by the nonstrange quarks of isospin $+\frac{1}{2}$ and $-\frac{1}{2}$ and by the strange quark, respectively (so that $e_u=+\frac{2}{3}$, $e_d=-\frac{1}{3}$, $e_s=-\frac{1}{3}$ for photon coupling), then the action of e_a is simple. For example, $e_a|uud\rangle = e_u|uud\rangle$, $e_a|duu\rangle = e_d|duu\rangle$, etc. We find immediately, for the SU_3 states of various symmetry [in each case for the state having the quantum numbers of the proton (e.g., p for octet, Δ^+ for decimet)],

$$\langle(\underline{8})_\alpha|e_a|(\underline{10})_S\rangle = \tfrac{1}{3}\sqrt{2}(e_u - e_d)\,,$$

$$\langle(\underline{8})_\alpha|e_a|(\underline{8})_\alpha\rangle = \tfrac{1}{3}(e_u + 2e_d)\,,$$

$$\langle(\underline{8})_\beta|e_a|(\underline{8})_\beta\rangle = e_u\,, \tag{A12}$$

$$\langle(\underline{8})_\beta|e_a|(\underline{10})_S\rangle = 0\,,$$

$$\langle(\underline{8})_\alpha|e_a|(\underline{8})_\beta\rangle = 0\,.$$

For the neutron, replace e_u by e_d and e_d by e_u.

For meson couplings in general, we should like to have e_a' replaced by a matrix, but it is very easy if we are coupling a neutral meson, say a π^0. Then, since $\pi^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}$, we obtain the correct result by substituting $e_u=+1$, $e_d=-1$ into (A12). Again, for η^0 coupling, since $\eta^0 = (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}$, substitute $e_u = 1/\sqrt{3}$, $e_d = 1/\sqrt{3}$, $e_s = -2/\sqrt{3}$ to get from (A12) the coupling of the η^0 to the proton. From these two cases, all the rest can be immediately obtained by using the standard Clebsch-Gordan (F, D) coefficients for SU_3. Thus, since in octets the proton-π^0 coupling is $F+D$ and the proton-η^0 coupling is $\sqrt{3}(F-\frac{1}{3}D)$, we can find (F, D) in the octet cases immediately:

$$\langle (\underline{8})_\alpha | e'_a | (\underline{8})_\alpha \rangle = (\tfrac{1}{6}, -\tfrac{1}{2}) ,$$
$$\langle (\underline{8})_\beta | e'_a | (\underline{8})_\beta \rangle = (\tfrac{1}{2}, \tfrac{1}{2}) . \tag{A13}$$

For the decimet it is only necessary to give one coupling, say proton-π^0, which in (A12) is $\frac{2}{3}\sqrt{2}$, since SU_3 determines all the other couplings. In Table VII we give a number of cases and the multiplicity R (explained in the text) that each matrix element squared must be multiplied by to get the total rate. Finally, for the singlet $(\underline{1})_A$, we give the matrix element from the singlet Λ state to a state of proton quantum numbers via a coupling with a K^+ meson,

$$\langle (\underline{8})_\alpha p | K^+ | (\underline{1})_A \Lambda \rangle = 0 ,$$
$$\langle (\underline{8})_\beta p | K^+ | (\underline{1})_A \Lambda \rangle = -1/\sqrt{3} . \tag{A14}$$

The factor for spin is the operator 1 or σ_z, σ_+, σ_- on the first quark. The matrix elements can be worked out directly from the states. We find the following matrix elements for the operators 1, and σ_z on the states of z component $+\frac{1}{2}$:

$$\langle \tfrac{1}{2} |_\alpha 1 | \tfrac{3}{2} \rangle_S = 0 ,$$
$$\langle \tfrac{1}{2} |_\alpha \sigma_z | \tfrac{3}{2} \rangle_S = +\tfrac{2}{3}\sqrt{2} ,$$
$$\langle \tfrac{1}{2} |_\alpha 1 | \tfrac{1}{2} \rangle_\alpha = 1 ,$$
$$\langle \tfrac{1}{2} |_\alpha \sigma_z | \tfrac{1}{2} \rangle_\alpha = -\tfrac{1}{3} ,$$
$$\langle \tfrac{1}{2} |_\beta 1 | \tfrac{1}{2} \rangle_\beta = 1 ,$$
$$\langle \tfrac{1}{2} |_\beta \sigma_z | \tfrac{1}{2} \rangle_\beta = 1 ,$$

a table immediately extended to the other operators σ_+, and σ_- and to other components of z spin, directly or via the Wigner-Eckart theorem.

Combining the spin and unitary spin, we list here in Table VIII the results for the product e_a and $e_a\vec{\sigma}$ between the various states of the SU_6 multiplets and a proton state of the fundamental octet, for the z component equal to $+\frac{1}{2}$. For cases with an octet, we give the matrix in the form of a coefficient times a parenthesis containing two numbers which are the values of F and D in the form (F, D). Thus,

TABLE VII. SU_3 matrix elements for decimet-octet transitions.

Transition	SU_3 matrix element	Weight R
$\Delta^+ \to N^+\pi^0$	1	$\frac{3}{2}$
$\to \Sigma^0 K^+$	1	$\frac{3}{2}$
$\to N^+\gamma$	$\frac{1}{2}$	1
$\Sigma^+ \to \Sigma^+\pi^0$	$\frac{1}{2}$	2
$\to \Sigma^+\eta^0$	$\frac{1}{2}\sqrt{3}$	1
$\to \Lambda^0\pi^+$	$\frac{1}{2}\sqrt{3}$	1
$\Sigma^0 \to N^+K^-$	$\frac{1}{2}$	2
$\to \Xi^- K^+$	$\frac{1}{2}$	2
$\Xi^- \to \Xi^-\pi^0$	$\frac{1}{2}$	3
$\to \Xi^-\eta^0$	$\frac{1}{2}\sqrt{3}$	1
$\to \Lambda^0 K^-$	$\frac{1}{2}\sqrt{3}$	1
$\to \Sigma^0 K^-$	$\frac{1}{2}$	3
$\Omega^- \to \Xi^0 K^-$	$\frac{1}{2}\sqrt{6}$	2

if a π^0 emission is wanted from excited proton to proton, the expression (F, D) is replaced by $(F+D)$, etc. For photons, replace (F, D) by $F + \frac{1}{3}D$ for protons and by $-\frac{2}{3}D$ for neutrons. In the case of decimets, the number given is directly the Δ^+ to π^0 and proton amplitude. To get other cases multiply by the matrix elements in Table VII. For the singlet, the number is the amplitude for Λ to K^- and proton. States $|\underline{70}_\beta\rangle$ and $|\underline{20}_A\rangle$ all have zero matrix elements.

It is also a simple matter to work out the matrices for the orbital motion. The operators here are

$$\langle g | (1, a_z, a_+, a_-) e^{-\lambda a_z} | n \rangle .$$

We find Table IX using the wave functions previously described. (All β states like β, $N=2$; $|2\rangle_\beta^2$, $|0\rangle_\beta^2$ give zero.) We finally combine all these to obtain the final matrix element. We illustrate this with a typical case. Suppose we wish to obtain the transition with π^0 emission from the $N(1700)$,

TABLE VIII. Matrix elements from octet proton with $S_z = +\frac{1}{2}$. The numbers in parentheses are (F, D) values.

Multiplet M	$\langle M, +\frac{1}{2} \lvert e_a \rvert P, +\frac{1}{2}\rangle$	$\langle M, +\frac{3}{2} \lvert e_a\sigma_{a+} \rvert P, +\frac{1}{2}\rangle$	$\langle M, +\frac{1}{2} \lvert e_a\sigma_{az} \rvert P, +\frac{1}{2}\rangle$	$\langle M, -\frac{1}{2} \lvert e_a\sigma_{a-} \rvert P, +\frac{1}{2}\rangle$
$^2(\underline{8})$ $\underline{56}$	$+\frac{1}{3}(+1, 0)$		$+\frac{1}{3}(+\frac{2}{3}, 1)$	$+\frac{1}{3}(+\frac{2}{3}, 1)$
$^4(\underline{10})$ $\underline{56}$	0	$-\frac{2}{9}\sqrt{6}$	$+\frac{4}{9}\sqrt{2}$	$+\frac{2}{9}\sqrt{2}$
$^2(\underline{1})$ $\underline{70}_\alpha$	$+\frac{1}{3}\sqrt{3}$		$+\frac{1}{3}\sqrt{3}$	$+\frac{1}{3}\sqrt{3}$
$^2(\underline{8})$ $\underline{70}_\alpha$	$+\frac{1}{2}(+\frac{1}{3}, 1)$		$+\frac{1}{6}(+\frac{5}{3}, 1)$	$+\frac{1}{6}(+\frac{5}{3}, 1)$
$^4(\underline{8})$ $\underline{70}_\alpha$	0	$-\frac{1}{6}\sqrt{3}\ (-\frac{1}{3}, 1)$	$+\frac{1}{3}(-\frac{1}{3}, 1)$	$+\frac{1}{6}(-\frac{1}{3}, 1)$
$^2(\underline{10})$ $\underline{70}_\alpha$	$+\frac{2}{3}$		$-\frac{2}{9}$	$-\frac{2}{9}$

TABLE IX. Matrix elements of space states.

State M	$\langle g\|1\|M,0\rangle$	$\langle g\|a_+\|M,+1\rangle$	$\langle g\|a_z\|M,0\rangle$	$\langle g\|a_-\|M,-1\rangle$
$N=0$ $L=0_s$	$+1$	0	0	0
$N=1$ $L=1_\alpha$	$-\lambda$	$+1$	$+1$	$+1$
$N=2$ $L=2_s$	$+\frac{1}{6}\sqrt{6}\,\lambda^2$	$-(\frac{1}{2})^{1/2}\lambda$	$-(\frac{2}{3})^{1/2}\lambda$	$-(\frac{1}{2})^{1/2}\lambda$
$N=2$ $L=2_\alpha$	$-\frac{1}{6}\sqrt{6}\,\lambda^2$	$+(\frac{1}{2})^{1/2}\lambda$	$+(\frac{2}{3})^{1/2}\lambda$	$+(\frac{1}{2})^{1/2}\lambda$
$N=2$ $L=0_s$	$-\frac{1}{6}\sqrt{3}\,\lambda^2$		$+(\frac{1}{3})^{1/2}\lambda$	
$N=2$ $L=0_\alpha$	$+\frac{1}{6}\sqrt{3}\,\lambda^2$		$-(\frac{1}{3})^{1/2}\lambda$	

which we identify as belonging to $^4(\underline{8})_{1/2}\,[\underline{70},1^-]$, going to a proton with $J_z=+\frac{1}{2}$. The $[\underline{70},1]$ is $(1/\sqrt{2})|\underline{70}_\alpha,1_\alpha\rangle+(1/\sqrt{2})|\underline{70}_\beta,1_\beta\rangle$, but the β pieces give zero matrix elements, so we have an overall $\sqrt{\frac{1}{2}}$ times the normal Clebsch-Gordan coefficients, to produce the correct angular momentum,

$$\langle{}^4\underline{8}_{1/2}[\underline{70},1^-],+\tfrac{1}{2}|\mathfrak{M}^P|p\rangle$$
$$=(\tfrac{1}{2})^{1/2}(\tfrac{1}{2})^{1/2}\langle{}^4\underline{8},+\tfrac{3}{2};1_\alpha,-1|\mathfrak{M}^P|p\rangle$$
$$-(\tfrac{1}{2})^{1/2}(\tfrac{1}{3})^{1/2}\langle{}^4\underline{8},+\tfrac{1}{2};1_\alpha,0|\mathfrak{M}^P|p\rangle$$
$$+(\tfrac{1}{2})^{1/2}(\tfrac{1}{6})^{1/2}\langle{}^4\underline{8},-\tfrac{1}{2};1_\alpha,+1|\mathfrak{M}^P|p\rangle\,.$$

Taking only the pieces of $\mathfrak{M}^P$ [Eq. (42)] which give nonzero matrix elements, this becomes (omit the factor $6F$)

$$\tfrac{1}{2}(+\sqrt{2}\,\beta)\langle{}^4\underline{8},+\tfrac{3}{2}|\sigma_+|p\rangle\langle 1_\alpha,-1|a_-^*|g\rangle$$
$$-(\tfrac{1}{6})^{1/2}(\beta-\gamma\lambda\delta)\langle{}^4\underline{8},+\tfrac{1}{2}|\sigma_z|p\rangle\langle 1_\alpha,0|a_z^*|g\rangle$$
$$-(\tfrac{1}{6})^{1/2}\gamma\langle{}^4\underline{8},+\tfrac{1}{2}|\sigma_z|p\rangle\langle 1_\alpha,0|1|g\rangle$$
$$+(\tfrac{1}{12})^{1/2}(-\sqrt{2}\,\beta)\langle{}^4\underline{8},-\tfrac{1}{2}|\sigma_-|p\rangle\langle 1_\alpha,+1|a_+^*|g\rangle\,.$$

Taking the appropriate numbers for these partial matrix elements from Tables VIII and IX, and noting that (F,D) is $(-\frac{1}{3},1)$, we find $F+D=\frac{2}{3}$ (for the π^0-coupling case) times

$$\tfrac{1}{2}(+\sqrt{2}\,\beta)\left(-\frac{\sqrt{3}}{6}\right)\times 1-(\tfrac{1}{6})^{1/2}(\beta-\gamma\lambda\delta)(+\tfrac{1}{3})\times 1$$
$$-(\tfrac{1}{6})^{1/2}\gamma(+\tfrac{1}{3})(-\lambda)+(\tfrac{1}{12})^{1/2}(-\sqrt{2}\,\beta)(+\tfrac{1}{6})\times 1$$
$$=\frac{1}{3\sqrt{6}}[\lambda\gamma(1+\delta)-3\beta]\,.$$

Multiplying the above by $\frac{2}{3}\times 6F$, this amplitude will be found under $S_{11}(1700)\to N\pi$ in Table II where the results of all similar calculations for the emission of pseudoscalar mesons are given. For a very few states, the final state is in the decimet instead of the proton. Obvious modifications are made, and new matrix elements supplement Tables VII and VIII in this case.

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